A Simple Introduction to Random Matrix Theory

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Historical Motivation

Figure: Eugene Wigner

Key Idea

When the system is too 'complicated' and has a large number of degrees of freedom, we can obtain some meaningful results by considering modelling the Hamiltonian as a matrix with random extries

Thank you for attending!


CHESS
100
LOL XD MAIN KAISE MAAN LU?


Bhai ne bola karne ka..
Matlab karne ka
PEPSI

RMT: For non believers

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Yeh lo proof

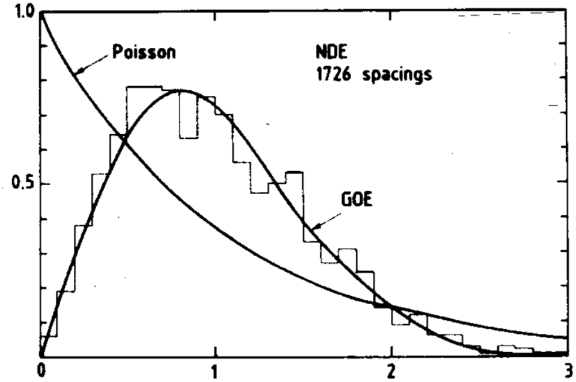
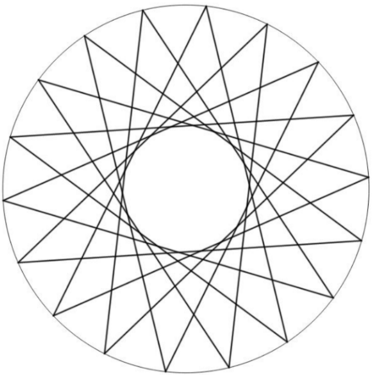
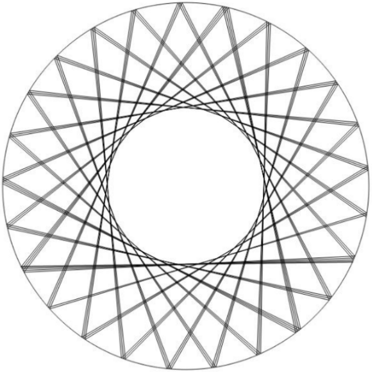
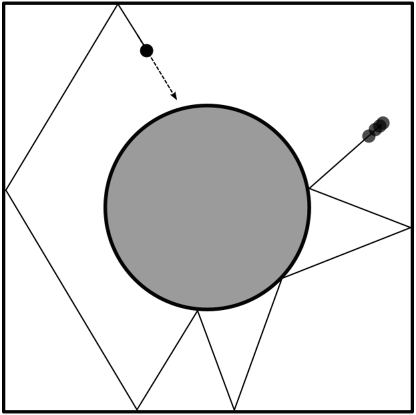


Figure: Nearest neighbour spacing experimental data versus theory

What is meant by Chaos?







- Okay, but what is Quantum Chaos?

- Berry-Tabor conjecture (Berry, Tabor 1977) If the corresponding classical dynamics is completely integrable, then P(s) exists and is equal to the waiting time between consecutive events of a Poisson process
- The BGS conjecture (Bohigas, Giannoni, Schmit 1984)
  Spectra of time–reversal invariant systems whose classical analogues are K systems show the same fluctuation properties as predicted by the GOE

The million dollar question

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{n^s} + \cdots$$

Hilbert–Pólya conjecture states that the non-trivial zeros of the Riemann zeta function correspond to eigenvalues of a self-adjoint operator.

Meet the Ensembles

$$H = \begin{bmatrix} e^{-H_{11}^2/2}/\sqrt{2\pi} & e^{-H_{12}^2/2}/\sqrt{2\pi} & \cdots \\ e^{-H_{21}^2/2}/\sqrt{2\pi} & e^{-H_{22}^2/2}/\sqrt{2\pi} & \cdots \\ \vdots & \vdots & e^{-H_{NN}^2/2}/\sqrt{2\pi} \end{bmatrix}$$

For real eigenvalues, we will symmetrize this matrix by $H_s = \frac{H+H^T}{2}$

$$\rho[H_s] = \prod_{i=1}^{n} e^{-H_{ii}^2/2}/\sqrt{2\pi} \prod_{i<j}^{n} e^{-H_{ij}^2}/\sqrt{\pi}$$

What is the jpdf of eigenvalues?

$$\rho(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_{N,\beta}} e^{-\frac{1}{2}\sum_{i=1}^{N} \lambda_i^2} \prod_{i<j}^{N} |\lambda_i - \lambda_j|^\beta$$

Where, $Z_{N,\beta}$ is the normalisation constant given by

$$Z_{N,\beta} = (2\pi)^{N/2} \prod_{j=1}^{N} \frac{\Gamma(1 + j\beta/2)}{\Gamma(1 + \beta/2)}$$

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

Wigner's surmise and semicircle law

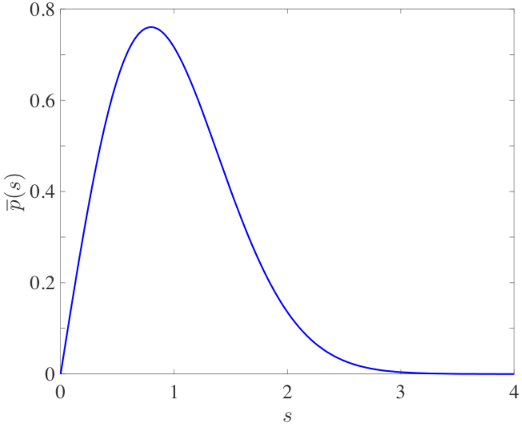
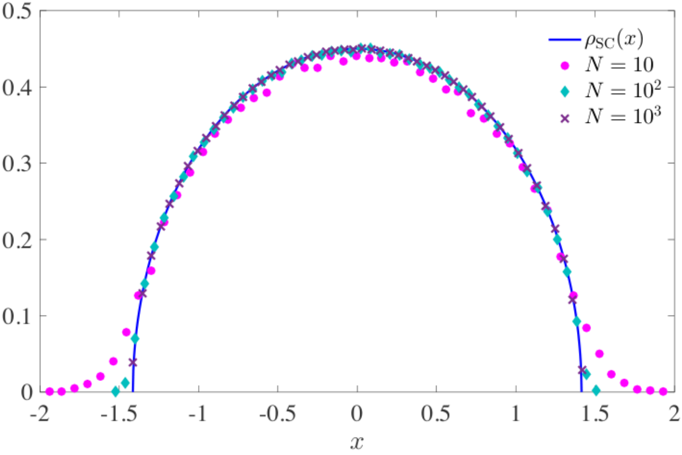
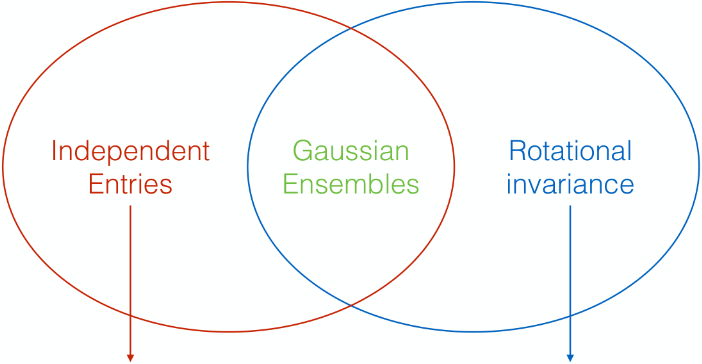


Figure: $\bar{P}(s) = \frac{\pi s}{2} e^{-\pi s^2/4}$



$$\lim_{N\to\infty} \sqrt{\beta N}\rho(\sqrt{\beta N}x) = \frac{1}{\pi}\sqrt{2-x^2}$$

Universal properties?



$$\rho[H] \propto \prod_{i=1}^{N} f_i(H_{ii}) \prod_{i<j} f_{ij}(H_{ij}) \qquad \rho[H] = \rho[UHU^{-1}]$$

Coulomb gas

Rescale the jpdf as $\lambda_i \to \sqrt{\beta N}\lambda_i$ and some simplification to obtain $Z_{N,\beta}$:

$$Z_{N,\beta} = C_{N,\beta} \int_{\mathbb{R}^N} \prod_{i=1}^{N} dx_i e^{-\beta N^2 \mathcal{V}[\lambda]}$$

$$C_{N,\beta} = (\sqrt{\beta N})^{N+\beta N(N-1)/2}$$

$$\mathcal{V}[\lambda] = \frac{1}{2N} \sum_{i=1} \lambda_i^2 - \frac{1}{2N^2} \sum_{i \neq j} ln|\lambda_i - \lambda_j|$$

- What about $\beta \neq 1, 2$ or 4 ? Dumitriu and Edelman

So What?

- RMT has a number of applications in Physics: The random matrix problem can be mapped onto a non-linear supersymmetric σ model (Efetov). It also has deep connections with to models of 2D Quantum Gravity, QCD.
- RMT has found several applications in other fields such as Information theory, Finance, etc.
- Personally, I enjoy studying RMT because it is fun and has some really 'cute' results.

Thank you!