① Motivation :- $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$.

The coupling constants can change due to RG flow with scale parameter μ. We can find fixed points so that they become scale invariant such as -

$$m = 0 \ , \lambda = 0$$

$$m = 0 \ , \lambda = \frac{16\pi^2 \varepsilon}{3}$$

Polschinki 1998 showed that if a theory has scale invariance, then it ~~can also have~~ also have conformal symmetry. Although Conformal group was studied for long but he proved it.

There is significance in Cosmological bootstrap, ~~Critical~~ Phase transition theory, & most importantly AdS/CFT which helps in Deep Inelastic scattering.

Conformal group on ②

Just general Action

$$[P_\mu, O(x)] = -i\partial_\mu O(x)$$

$$[M_{\mu\nu}, O(x)] = \left(i(x_\mu\partial_\nu - x_\nu\partial_\mu) + iS_{\mu\nu} \right) O(x)$$

$$[D, O(x)] = \left(+ix^\nu\partial_\nu + i\Delta \right) O(x)$$

$$[K_\mu, O(x)] = \left(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 2x^\nu S_{\mu\nu} + 2x_\mu\Delta \right) O(x)$$

② Global Conformal Invariance ~~In~~ Eu

Conformal group :- In d dimension, Eucledian space

$$x \to x'$$

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x) \quad \text{— ①}$$

For Poincare group $\Lambda(x) = 1$

Conformal group preserves angles.

$$x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x).$$

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$$ } Transformation properties of Tensors like $\int dx f(x) = \int \frac{\partial x}{\partial y} f(y) dy$

$$= (\delta^\alpha_\mu - \partial_\mu \varepsilon^\alpha)(\delta^\beta_\nu - \partial_\nu \varepsilon^\beta) g_{\alpha\beta}$$

$$= g_{\mu\nu} - (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu)$$

① says. $\boxed{\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = f(x) g_{\mu\nu}}$ — ②

Taking trace $\quad \partial^\mu \varepsilon_\mu + \partial^\mu \varepsilon_\mu = f(x) d$

$$\boxed{f(x) = \frac{2}{d} \partial_\rho \varepsilon^\rho}$$

Diff. ∂_ρ on ② & permuting

$$2 \partial_\mu \partial_\nu \varepsilon_\rho = \eta_{\mu\rho} \partial_\nu f + \eta_{\nu\rho} \partial_\mu f - \eta_{\mu\nu} \partial_\rho f$$

Contracting with $\eta^{\mu\nu}$ $\Rightarrow$ $\quad 2\partial^2 \varepsilon_\mu = (2-d) \partial_\mu f$

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Combining we get

$$(2-d)\,\partial_\mu \partial_\nu f = \eta_{\mu\nu}\partial^2 f$$

Contracting $\eta^{\mu\nu}$

$$(d-1)\,\partial^2 f = 0$$

For $d>2$, f is at most linear

$$f(x) = A + B_\mu x^\mu$$

Then from ∂_μ 2, we see that ε_μ is at most quadratic

$$\varepsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \,, \quad c_{\mu\nu\rho} = c_{\mu\rho\nu}$$

Putting in 2, $\quad b_{\mu\nu} + b_{\nu\mu} = \dfrac{2}{d} b^\lambda{}_\lambda \eta_{\mu\nu}$

$$b_{\mu\nu} = \underbrace{\alpha\,\eta_{\mu\nu}}_{\text{Scaling}} + \underbrace{m_{\mu\nu}}_{\text{Rotation}} \,, \quad m_{\mu\nu} = -m_{\nu\mu}$$

Quadratic term gives $\quad c_{\mu\nu\rho} = \eta_{\mu\rho} b_\nu + \eta_{\mu\nu} b_\rho - \eta_{\nu\rho} b_\mu \,, \quad b_\mu = \frac{1}{d} c^\sigma{}_{\sigma\mu}$

$$x'^\mu = x^\mu + 2(x\cdot b)x^\mu - b^\mu x^2 \qquad \text{Exponentiating gives}$$

Translation	$x'^\mu = x^\mu + a^\mu$
Dilation	$x'^\mu = \alpha x^\mu$
Rigid Rotation	$x'^\mu = M^\mu{}_\nu x^\nu$
Special Conformal Transf.	$x'^\mu = \dfrac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}$

4) ~~We have to exponentiate these transformation~~

SCT can be reexpressed as

$$\frac{x'^{\mu}}{x'^{2}} = \frac{x^{\mu}}{x^{2}} - b^{\mu}$$

SCT is nothing but a translation, preceded & followed by an inversion $x^{\mu} \to x^{\mu}/x^{2}$.

$$0 \rightleftarrows \infty$$

(5) ~~By~~ Generators :- $\Phi^{0}(x') \to \Phi'(x') = U\,\Phi(x)$

U is the group's action on $\Phi(x)$

we can ~~from~~ take infinitisimal limit to say that

$$U = 1 \mp i \underbrace{\omega_a}_{\text{Infinitisimal parameter}} \underbrace{T^a}_{\text{Generators of group}}$$

We start by Translation, for fields action is shown as

$$[P^\mu, \Phi(x)] = -i\partial^\mu \Phi(x)$$

This is called adjoint representation.
Similar to Heisenberg eqn. $\boxed{[H, O] = i\partial_t O}$

There is a more technical way to get it using word identity but I won't go there.

for Poincare group, Rotation we have ~~as~~ $M_{\mu\nu}$ acting on local operator

$$[M_{\mu\nu}, O^a(0)] = \underbrace{(S_{\mu\nu})^a{}_b}_{\text{Spin operator}}\, O^b(0)$$

$\to$ spin indices

We can translated the operator as $e^{iPx} M_{\mu\nu} e^{-iPx}$

Then using Baker Campbell formula

$$e^X e^Y = \exp\left[X + Y + \frac{1}{2}[X,Y] + \dots\right]$$

(6) we can get
$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + S_{\mu\nu}$$

Similarly for D & K_μ we have at $x=0$
$$[D, O(0)] = -i\Delta O(0) \qquad [K_\mu, O(0)] = k_\mu O(0)$$

Translating them gives
$$e^{ixP} D e^{-ixP} = D + x^\nu P_\nu$$
$$e^{ixP} K_\mu e^{-ixP} = K_\mu + 2x_\mu D - 2x^\nu L_{\mu\nu} + 2x_\mu (x^\nu P_\nu) - x^2 P_\mu$$

Then,
$$[D, \Phi(x)] = (-ix^\nu \partial_\nu \mp i\Delta)\Phi(x)$$
$$[K_\mu, \Phi(x)] = \{k_\mu \mp 2x_\mu i\Delta - x^\nu S_{\mu\nu} - 2ix_\mu x^\nu \partial_\nu + ix^2 \partial_\mu\} \Phi(x)$$

Using Schur's lemma we also know $k_\mu = 0$

~~Then~~ algebra ~~is~~ of any group is
$$[T^a, T^b] = i \underbrace{f_{abc}}_{\text{structure constants}} T^c$$

we have Conformal algebra;
$$[D, P_\mu] = iP_\mu$$
$$[D, K_\mu] = -iK_\mu$$
$$[K_\mu, P_\nu] = 2i(\eta_{\mu\nu} D - L_{\mu\nu})$$
$$[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu)$$
$$[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu)$$
$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho})$$

9 Primaries & Descendants : Most important ~~rel~~ relations are

$$[D, P_\mu] = i P_\mu$$
$$[D, K_\mu] = -i K_\mu$$

$$D P_\mu O(0) = ([D, P_\mu] + P_\mu D) O(0) = (i P_\mu + i \Delta P_\mu) O(0)$$
$$= (\Delta + 1) i P_\mu O(0)$$

~~So~~ similarly we have

$$D K_\mu O(0) = (\Delta - 1) i K_\mu O(0)$$

So, P^μ ~~can~~ ascend the operators & ~~is~~ like creation operator
K^μ descend in states of D.

~~So~~, We can't go down forever then choosing some operators

s.t. $K_\mu O(0) = 0$, with $D O(0) = \Delta$

~~Then~~ then, we can form $P_\mu O$, $P_\mu P_\nu O'$, $P^3_{\kappa\mu} O$
$\Delta+1$, $\Delta+2$, $\Delta+3$

These O's are called primaries & $P_\mu O$ ...etc are descendants.

8) Correlation functions: Choosing conformally invariant vacuum w. $D|0\rangle = 0$

Two point functions: Then applying D on two point functions

$$\langle 0|[D, O_i(x_i) O_j(x_j)]|0\rangle = \langle 0|[D, O_i(x_i)] O_j(x_j)|0\rangle + \langle 0| O_i(x_i)[D, O_j(x_j)]|0\rangle$$

$$= (x_i^\mu \partial_\mu^i + \Delta_i + x_j^\mu \partial_\mu^j + \Delta_j) \langle O_i(x_i) O_j(x_j)\rangle$$

Rotational & Translational invariance says that

$$\langle O_i(x_i) O_j(x_j)\rangle \propto F(|x_i - x_j|)$$

Then choosing $x_j = 0$ & $x_i = x$

$$(x\cdot\partial + \Delta_i + \Delta_j)\langle O_i(x) O_j(0)\rangle = 0$$

(because of invariant vacuum)

This eq^n gives form

$$\langle O_i(x) O_j(0)\rangle = \cancel{\frac{C}{|x_i - x_j|^{\Delta_1+\Delta_2}}} \; \frac{C}{|x|^{\Delta_1+\Delta_2}}$$

More generally

$$\langle O_i(x_i) O_j(x_j)\rangle = \frac{C}{|x_i - x_j|^{\Delta_1+\Delta_2}}$$

Using SCT we also get condition $\delta_{\Delta_1,\Delta_2}$

$$\langle O_i(x_i) O_j(x_j)\rangle = \frac{C\,\delta_{\Delta_1\Delta_2}}{|x_i - x_j|^{\Delta_1+\Delta_2}}$$

Then $\boxed{\langle O(x_i) O(x_j)\rangle = \frac{C}{|x_i - x_j|^{2\Delta}}}$

(7) Three point functions :- Rotation & Translation forces & also dilation

$$\langle O_1(x_1) O_2(x_2) O_3(x_3) \rangle = \frac{C_{abc}}{|x_1 - x_2|^a |x_2 - x_3|^b |x_1 - x_3|^c}$$

with constraint $a+b+c = \Delta_1 + \Delta_2 + \Delta_3$

Then SCT gives

$$\boxed{\langle O_1(x_1) O_2(x_2) O_3(x_3) \rangle = \frac{C_{123}}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3} |x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1} |x_3 - x_1|^{\Delta_3+\Delta_1-\Delta_2}}}$$

Four point function :- It's not possible to constrain four point function. But we can form it in terms of cross ~~re~~ ratios which are invariants

$$u = \frac{x_{12}\, x_{34}}{x_{13}\, x_{24}}, \quad v = \frac{x_{12}\, x_{34}}{x_{23}\, x_{14}}$$

Then,

$$\boxed{\langle O_1(x_1) O_2(x_2) O_3(x_3) O_4(x_4) \rangle = \prod_{i<j}^{4} x_{ij}^{\Delta/3 - \Delta_i - \Delta_j} f(u,v)}$$

Now you might think CFT is not that powerful But we can do much more than this using something called OPE

OPE Expansion :-

Let $O_1(x_1) O_2(x_2)$, ~~are~~ here very close or $x_1 \to x_2$ & there is no other operator nearby.

Then we can show that this product can be written as sum of primary operators & descendants.

$$\lim_{x_1 \to 0} \boxed{O_1(x_1) O_2(0) |0\rangle = \sum_k' C_{ijk}(x, P) O_k(0)|0\rangle}$$

or

$$\lim_{x_1 \to x_2} \boxed{O_i(x_1) O_j(x_2)|0\rangle = \sum_k' C_{ijk}(x_{12}, \partial_2) O_k(x_2)|0\rangle}$$

Now multiply with $O_{k'}(x_3)$ & take expectation value

$$\lim_{x_1 \to x_2} \langle O_i(x_1) O_j(x_2) O_{k'}(x_3)\rangle = \sum_k' C_{ijk}(x_{12}, \partial_2) \langle O_k(x_2) O_{k'}(x_3)\rangle$$

which we know

$$\frac{f_{ijk}}{|x_{12}|^{\Delta_i+\Delta_j-\Delta_k} |x_{23}|^{\Delta_j+\Delta_k-\Delta_i} |x_{31}|^{\Delta_k+\Delta_i-\Delta_j}} = C_{ijk}(x_{12}, \partial_2) \frac{1}{|x_{23}|^{2\Delta_k}}$$

Now we can expand in terms of $|x_{12}| \to 0$ & $|x_{23}| > |x_{12}|$

then we can fix C_{ijk} upto constant f_{ijk}

This can use to constraint any N-point function

$$\langle \underbrace{O_1 O_2}O_3 O_4\rangle = \sum_{O,O'}' f_{O_1 O_2 O} f_{O' O_3 O_4} C_a(x_{12}, \partial_2) C_b(x_{34}, \partial_4) \langle O^a(x_2) O'^b(x_4)\rangle$$