Quantum Error Correction Codes - Stabilizer Formalism

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Motivation

Information is Physical

∼ Rolf Landauer

All forms of computing require the interaction of some physical entity with the environment. *Environment is Noisy!*

Our information keeps changing, and thus, we need to devise a way to correct the errors. All errors occur with some probability and we can correct errors with maximum probability considering the error model in known.

Quantum Error Correction

Noise

The Quantum Channel

A physical communication Channel that can transmit quantum information, i.e., the state of a qubit. It can be considered as a *CPTP* operator.

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger \quad s.t. \quad \sum_i E_i E_i^\dagger = \mathbb{1}$$

Errors

Bit-Flip Error : $|0\rangle \longleftrightarrow |1\rangle$

Phase-Flip Error : $|i\rangle \longleftrightarrow (-1)^i |i\rangle$

Idea

Redundancy : We encode k qubits to $n > k$ qubits and then pass the new states through the quantum channel. Do some sort of *Majority Voting* to eliminate errors.

Quantum Error Correction

Theorem

No Cloning Theorem: Creating an independent and identical copy of an arbitrary unknown quantum state is impossible.

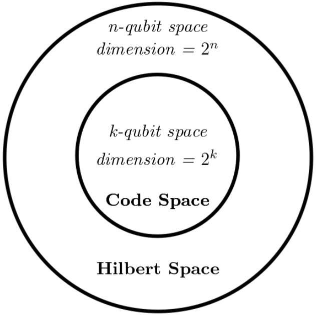
$$|\psi\rangle \otimes |s\rangle \quad \textit{can't go to} \quad |\psi\rangle \otimes |\psi\rangle$$

Error Correcting Code

An encoding scheme that transmits messages in such a way that the message can be recovered even if some bits are error full.

Code & Codewords

Code is a subspace of a larger Hilbert space.
Codeword is any vector that lies within the code.

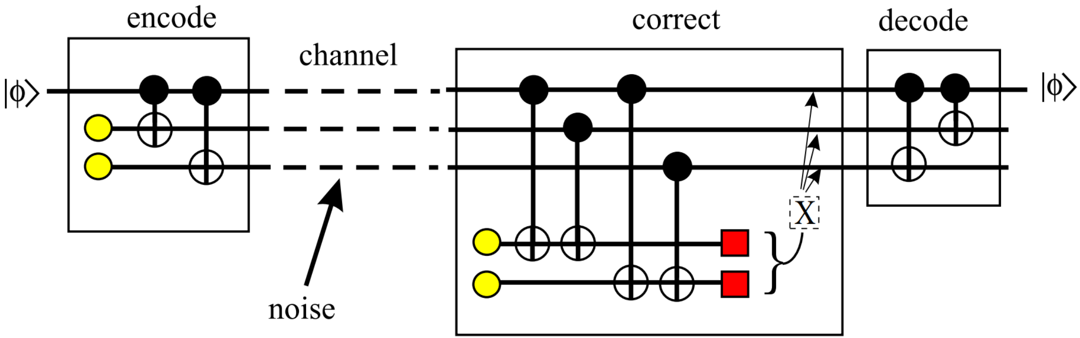


Shor Code

3 Bit Code

Let's encode each bit to 3 bits such that

$$a|0\rangle + b|1\rangle \xrightarrow{encoding} a|000\rangle + b|111\rangle$$



Encoded State	Syndrome Measurement	Probability
$a\|000\rangle + b\|111\rangle$	$a\|000\rangle + b\|111\rangle\|00\rangle$	$(1-p)^3$
$a\|100\rangle + b\|011\rangle$	$a\|100\rangle + b\|011\rangle\|11\rangle$	$p(1-p)^2$
$a\|010\rangle + b\|101\rangle$	$a\|010\rangle + b\|101\rangle\|10\rangle$	$p(1-p)^2$
$a\|001\rangle + b\|110\rangle$	$a\|001\rangle + b\|110\rangle\|01\rangle$	$p(1-p)^2$
$a\|110\rangle + b\|001\rangle$	$a\|110\rangle + b\|001\rangle\|01\rangle$	$p^2(1-p)$
$a\|101\rangle + b\|010\rangle$	$a\|101\rangle + b\|010\rangle\|10\rangle$	$p^2(1-p)$
$a\|011\rangle + b\|100\rangle$	$a\|011\rangle + b\|100\rangle\|11\rangle$	$p^2(1-p)$
$a\|111\rangle + b\|000\rangle$	$a\|111\rangle + b\|000\rangle\|00\rangle$	p^3

Encoding

Phase Flip

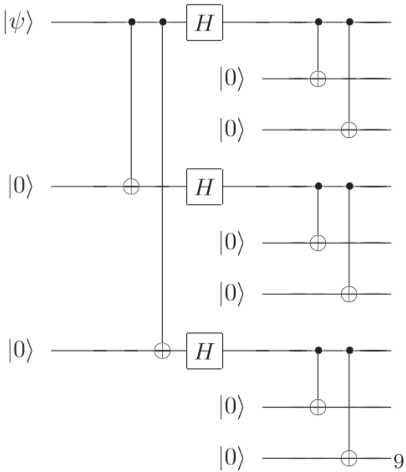
$$a|0\rangle + b|1\rangle \rightarrow a|+++\rangle + b|---\rangle$$

How to do both?

1. Phase Flip Encoding
2. Bit Flip Encoding

$$|0\rangle \rightarrow \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \rightarrow \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$



Syndrome Measurement

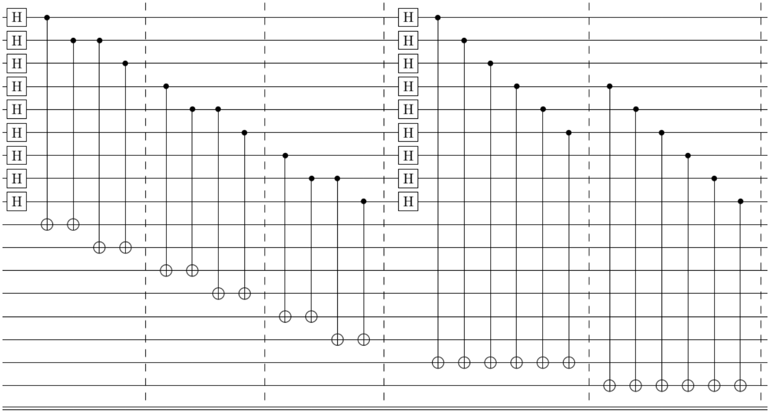


Figure 2: Shor Code Syndrome Extraction

Stabilizer Formalism

Errors and States

Pauli Error Group

$$G_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$$

$$G_1 = \langle i, X, Z \rangle$$

For n qubits, errors would be n - *fold tensor product* of elements of G.

Properties of Pauli Matrices:

$$\{A, B\} = 0 \quad \textbf{or} \quad [A, B] = 0 \quad \forall\ A, B \in G_n$$

Consider a state $|\psi\rangle$ and an error M such that $M|\psi\rangle = |\psi\rangle$.

Now, for $E \in G_n$ such that $E, M = 0$;

$$ME|\psi\rangle = -EM|\psi\rangle = -|\psi\rangle$$

Stabilizer Subgroup

Stabilizer Subgroup

Suppose S is a subgroup of G_n; we define V_S as the set of n *qubit states* fixed by every element of S. V_S is called the vector space stabilized by S and S is called the *stabilizer* of the space V_S .

Note that $\forall g_i \in G_n, \quad g_i^2 = 1 \quad \text{and} \quad Tr[g_i] = 0$
Thus, g_i has +1 and -1 eigenvalues each with degeneracy 2^{n-1}
Take $P_1 = \frac{I+g_1}{2}$ as projector onto +1 eigenspace of g_1 and observe that $Tr[P_1 g_2] = 0$
i.e. it again divide eigenspace in half.

Now, $S = \langle g_1, \dots, g_r \rangle \implies |S| = 2^{n-r} = 2^k$
Hence, $r = n - k$, i.e., $\exists \quad n - k$ generators for the stabilizer subgroup.

Normalizer Subgroup

Normalizer Subgroup

The set of elements in G that fixes S under conjugation, i.e., $N(S) = \{h|hS = Sh\}$

Note that for $A \in N(S)$, $M \in S$, and $|\psi\rangle$ in V_S;

$$A^\dagger M A = \pm M A^\dagger A = \pm M$$

Since $-\mathbb{1} \notin S$, if $A \in N(S)$, then $[A, M] = 0$

$$\therefore \quad MS|\psi\rangle = AM|\psi\rangle = A|\psi\rangle \implies A|\psi\rangle \in V_S$$

Now, $|S||N(S)/S||G_n/N(S)| = |G_n| \implies 2^{n-k}|N(S)/S|2^{n-k} = 4 \cdot 4^n$

$$\therefore \quad |N(S)| = 4 \cdot 2^{n+k}; \quad \#\text{generators} = n + k$$

Visualization

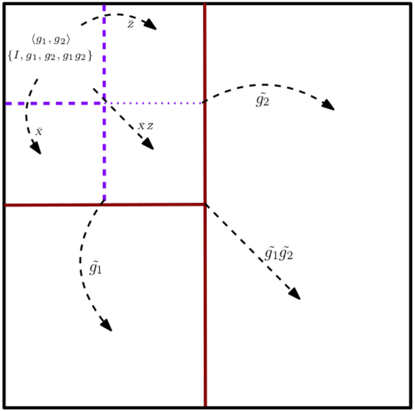


Figure 3: [3,1] Code Operator Space

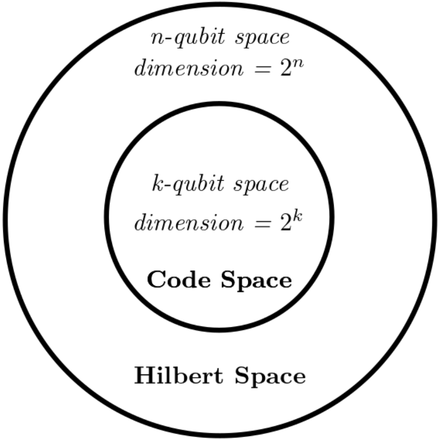


Figure 4: Corresponding Hilbert & Code

Detection and Correction

Theorem

Let S be the stabilizer for a stabilizer code $C(S)$. Suppose $\{E_j\}$ is a set of operators in G_n such that $E_j^{\dagger} E_k \notin N(S) - S \; \forall j, k$. Then $\{E_j\}$ is a correctable set of errors for code $C(S)$.

For errors that are correctable, measure the generators g_1 through g_{n-k} and obtain the error syndrome with corresponding results β_1 through β_{n-k}.

Suppose error E_j occur on qubit j, and the syndrome result is β_l corresponding to g_l.

- If E_j is the unique error operator with obtained syndrome, we can recover the state by applying $E_j^{\dagger}$.
- If E_j and $E_{j'}$ both have the same obtained error syndrome, $E_j P E_j^{\dagger} = E_{j'} P E_{j'}^{\dagger}$ implies that applying $E_j^{\dagger}$ will recover the state.

Shor Code & Five Qubit Code

Shor Code Stabilizers

g_1	$ZZIIIIIII$
g_2	$IZZIIIIII$
g_3	$IIIZZIIII$
g_4	$IIIIZZIII$
g_5	$IIIIIIZZI$
g_6	$IIIIIIIZZ$
g_7	$XXXXXXIII$
g_8	$IIIXXXXXX$
$\bar{X}$	$XXXXXXXXX$
$\bar{Z}$	$ZZZZZZZZZ$

Five Qubit Code Stabilizers

g_1	$XZZXI$
g_2	$IXZZX$
g_3	$XIXZZ$
g_4	$ZXIXZ$
$\bar{X}$	$XXXXX$
$\bar{Z}$	$ZZZZZ$

Conclusion

General Codes

In summary, *Stabilizer Code* can correct all errors other than *logical errors*.

Entanglement Assisted Codes

Given any set of operators, we use entanglement as a resource to convert them into stabilizers. Entanglement Assisted Codes allow us to choose operators which are better suited for measurement as per our system.

Subsystem Codes

Using the known properties of the physical system, we can avoid some errors as well as correct other errors. It uses idea of noiseless subsystems and break the normaliser into logical and gauge group where gauge group is equivalent to no noise.

Fault-Tolerant Computation

Quantum Error Correction doesn't only protect stored or transmitted quantum information, it should also protect it as the information dynamically undergoes computation.

Fault Tolerance ensures good computation be achieved even with faulty logic gates, provided only that the error probability per gate is below a certain *threshold*.

If one component in the procedure fails, then it causes at most one error in each encoded block of qubits output.