Measuring Earth's Motion Using a Population of Gravitational-Wave Sources

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Introduction and Motivation

- Cosmological principle– Isotropy
- Test of isotropy– Cosmic Microwave Background Radiation (CMB)

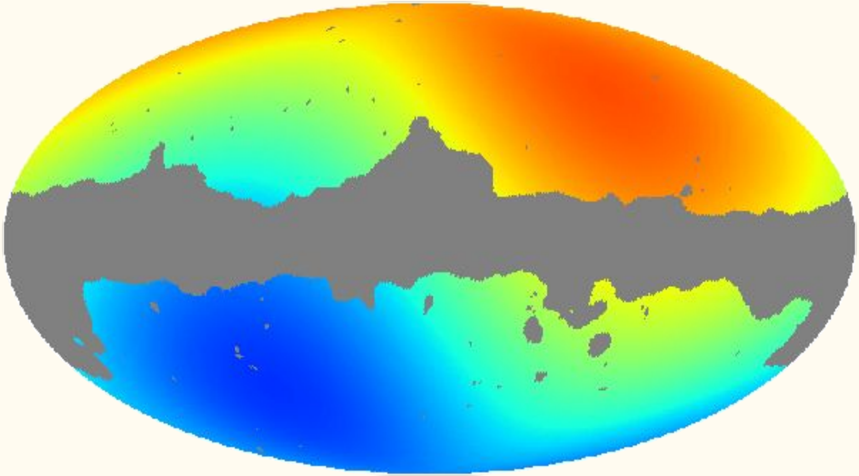
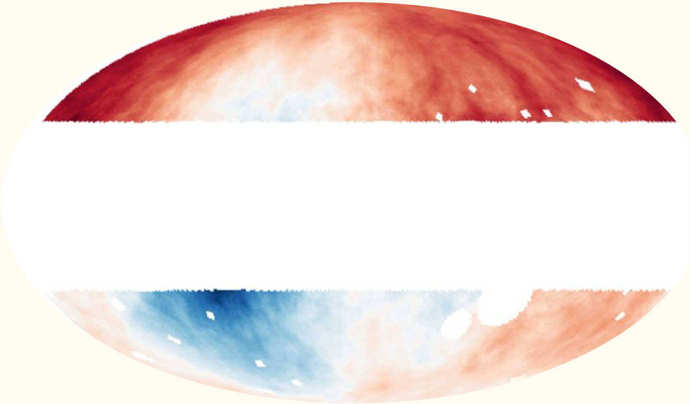


Image Credit: Planck Collaboration (Planck 2013 results. XXVII)

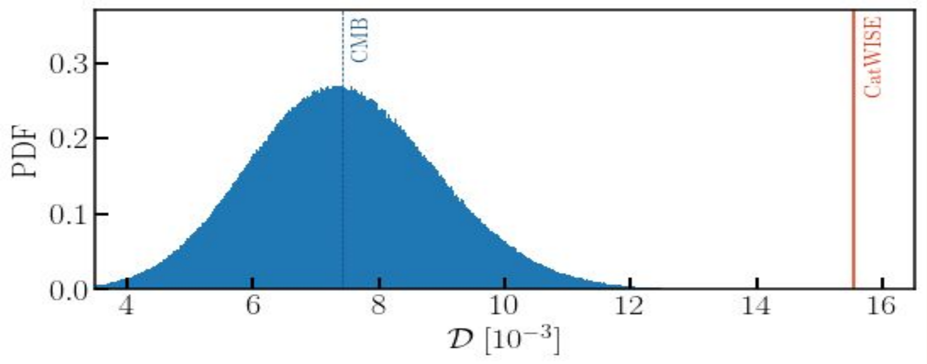
- Dipole anisotropy observed in CMB!
- Interpreted in terms of the earth's motion w.r.t. *cosmic frame of rest*

- Dipole anisotropy also detected in large scale structure (e.g. distribution of quasars, diffuse X-ray background, radio sources)
- Quasar observations (eg. Secrest et. al 2021) show a dipole amplitude over twice as large than CMB dipole



Density map of CatWISE quasar sample

Image Credit: Secrest et.



Amplitude of the dipole in the CatWISE quasar sample vs. the expectation from CMB studies

Image Credit: Secrest et. al 2021

- Can gravitational waves resolve this tension?

What is the cosmic rest frame, and why are we moving relative to it?

- The cosmic rest frame is a frame in which the CMB (and large scale structure) appears isotropic. It is the frame that is comoving with the expansion of the universe
- The milky way is gravitating towards the the so-called great attractor
- The solar system is in orbit around the galactic centre
- Earth orbits the sun, but this motion is relatively small and only serves to periodically modulate the larger motion towards the great attractor

Effects of relative motion on sky distribution of GW sources

- Relativistic effects– doppler boosting and aberration– result in a dipolar sky distribution of observed sources
- Merging compact binaries have a characteristic chirp mass that is distributed isotropically on the sky in the rest frame
- Chirp Mass is related to the frequency of the GW signal and hence gets redshifted due to doppler effects
- Because the doppler redshift depends on the location of the event on the sky, the observed mass distribution no longer remains isotropic

Effect of relative motion on source distribution

- Can we come up with a model for the distribution of the observed sky location of GW events?

Relativistic beaming: $d\Omega_{rest} = d\Omega_{obs}\Delta^2$

Rest-frame distribution: $\frac{dN_{rest}}{d\Omega_{rest}} = k$

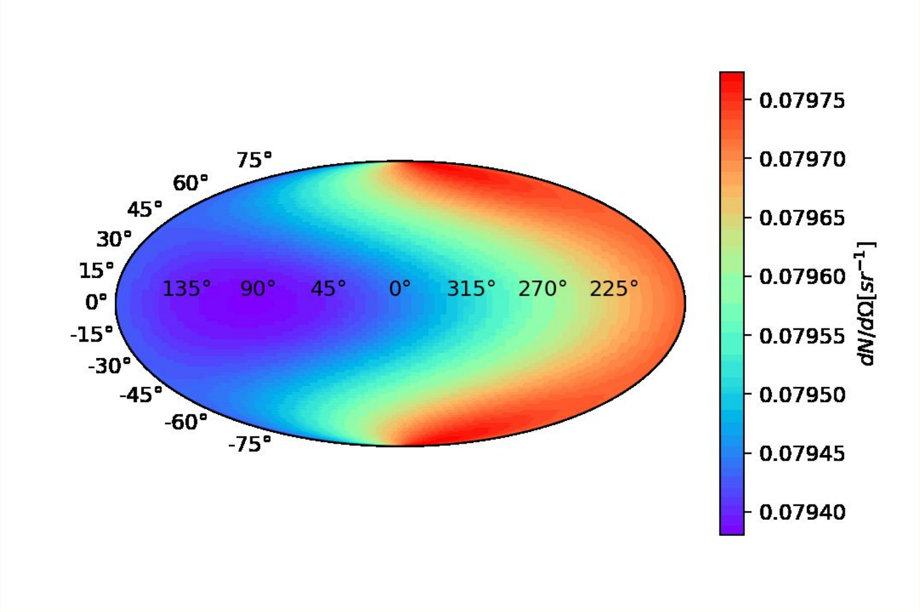
Observed distribution: $\frac{dN_{obs}}{d\Omega_{obs}} = k\Delta^2$

Doppler factor:

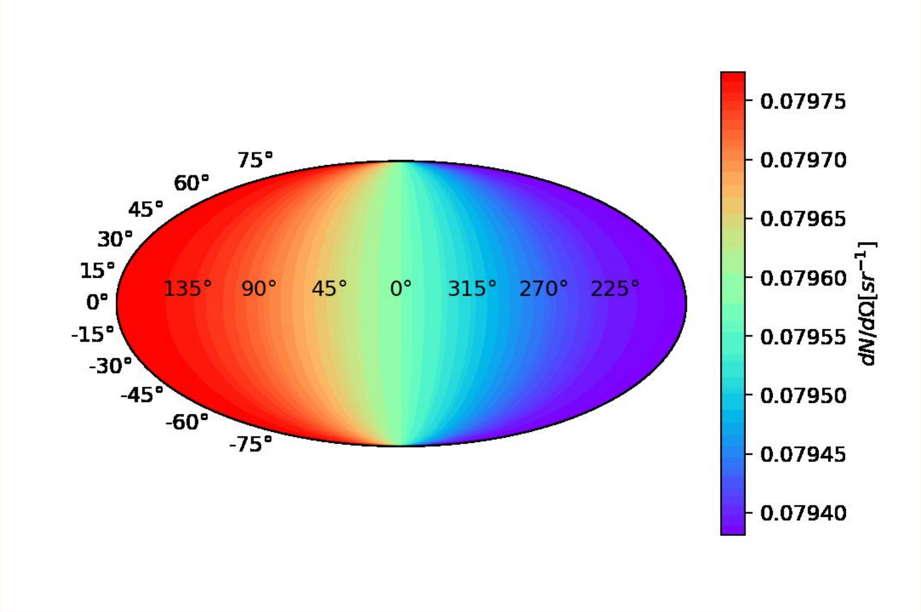
$$\Delta \approx \left(1 + \frac{v}{c}\cos\Theta\right)$$

$$\Delta^2 \approx \left(1 + \frac{2v}{c}\cos\Theta\right)$$

$$\cos\Theta = \cos\delta\cos\delta_v\cos(\phi - \phi_v) + \sin\delta\sin\delta_v$$



Fixed declination at 48° (CMB value)



Fixed right ascension at 264° (CMB value)

Animation showing the expected distribution of sources an observer would measure if earth were moving at a speed of 370 Km/s (CMB value) along different directions, assuming an isotropic distribution in the cosmic rest frame

Effect of relative motion on mass distribution

Relativistic beaming: $d\Omega_{rest} = d\Omega_{obs}\Delta^2$

$$\mathcal{M}^c_{\rm obs} \propto \left(\nu_{\rm obs}^{-11/3}\dot{\nu_{\rm obs}}\right)^{3/5}$$

Doppler effect:
$$m_{rest} = m_{obs}\Delta$$
$$dm_{rest} = dm_{obs}\Delta$$

Doppler factor:

$$\Delta \approx \left(1 + \frac{v}{c}\cos\Theta\right)$$
$$\Delta^2 \approx \left(1 + \frac{2v}{c}\cos\Theta\right)$$

Rest-frame distribution: $$\frac{d^2N_{rest}}{d\Omega_{rest}dm_{rest}} = k\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\left[\frac{(m_{rest}-m_0)^2}{2\sigma^2}\right]}$$

Observed distribution: $$\frac{d^2N_{obs}}{d\Omega_{obs}dm_{obs}} = k\Delta^2\frac{1}{\sqrt{2\pi(\sigma/\Delta)^2}}e^{-\left[\frac{(m_{obs}-m_0/\Delta)^2}{2(\sigma/\Delta)^2}\right]}$$

Animation showing the mass distribution of sources an observer would measure if earth were moving at a speed of 300 Km/s along different directions, assuming a gaussian distribution in the cosmic rest frame

Methods

Numerical Experiments

- Simulate mock gravitational wave events distributed in a dipolar fashion on the sky by randomly sampling points from a dipole distribution with particular velocity hyper-parameters. Assign a Gaussian random number as the chirp mass to each event
- Obtain the joint hyper-posterior probability distribution for the injected parameters using a hierarchical Bayesian inference formalism
- Sample the hyper-posterior using Markov Chain Monte Carlo (MCMC) to obtain best-fit values and uncertainties for the parameters
- Monitor the recovery of the velocity parameters as a function of the number of simulated events

A Back-of-the-envelope Calculation

- Can we estimate how many events would be needed to detect a dipole?
- Poisson noise in random sampling: $\sim \frac{1}{\sqrt{N}}$

  Dipole anisotropy: $\frac{v}{c} \sim 10^{-3}$

  $\implies N \gtrsim 10^6$
- We need at least a million events!

Hierarchical Bayesian Inference

- A framework to study the properties of a *population*
- *Hyper-parameters* describe a model for the population distribution of a property of interest (the *hypermodel*)
- Obtain posterior distribution of hyper-parameters (the *hyper-posterior*) in terms of the population hyper-model
- A hierarchy of inference:

parameter estimation → population inference

Details of Hierarchical Inference

Bayes theorem:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Re-write:

$$p(\theta|d, M) = \frac{\mathcal{L}(d|\theta)\pi(\theta|M)}{\mathcal{Z}_M}$$

d : GW event data
θ : event parameters
M : model
p : posterior
$\mathcal{L}$: likelihood
π : prior
$\mathcal{Z}_M$: evidence

Hyper-likelihood for the hyper-parameters $\vec{v}$ describing the model

$$\mathcal{L}(d|\vec{v}) = \int \mathcal{L}(d|\theta)\pi(\theta|\vec{v})d\theta = \mathcal{Z}_{\vec{0}} \int p(\theta|d,\vec{0})\frac{\pi(\theta|\vec{v})}{\pi(\theta|\vec{0})}d\theta$$

which can be approximated by 'recycling' samples from the posterior distribution of the parameters obtained using the isotropic model

$$\mathcal{L}(d|\vec{v}) = \frac{\mathcal{Z}_{\vec{0}}}{n} \sum_{k=1}^{n} \frac{\pi(\theta^k|\vec{v})}{\pi(\theta^k|\vec{0})}$$

Suppose we have a dataset $\{d_i\}$ for N independent events

The hyper-posterior is given by

$$p(\vec{v}|\{d_i\}) = \frac{\mathcal{L}_{tot}(\{d_i\}|\vec{v})\pi(\vec{v})}{\mathcal{Z}_{\vec{v}}^{tot}}$$

where

$$\mathcal{L}_{tot}(\{d_i\}|\vec{v}) = \prod_{i=1}^{N} \mathcal{L}(d_i|\vec{v})$$

Numerical Experiments-I

Hyper-model for sky distribution

$$\pi(\delta, \alpha|\vec{v}) \propto \frac{d^2N}{d\delta d\alpha} = \cos\delta \frac{dN}{d\Omega}$$

Normalised PDF:

$$\pi(\delta, \alpha|\vec{v}) = \frac{\cos\delta}{4\pi}\Delta^2$$

Celestial Coordinates:
Right ascension : δ
Declination : α
Event parameters : $\theta \equiv \{\delta, \alpha\}$
Hyper-parameters : $\vec{v} \equiv \{v, \delta_v, \alpha_v\}$

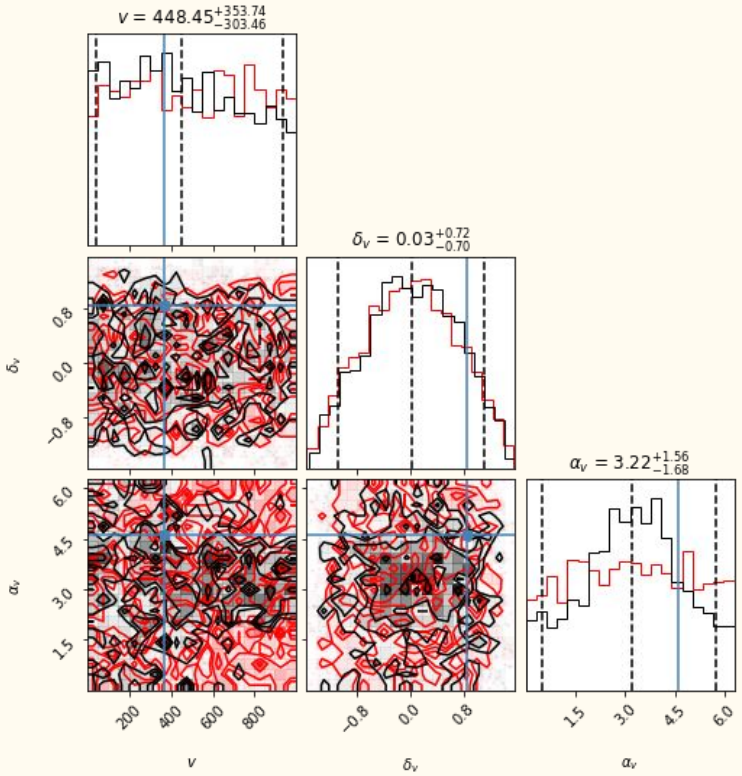
Assumptions

- The sources for all events are perfectly localised (point sources)
- There are no selection effects, ie., the 'detectors' are equally sensitive to all sky locations and all masses at all times
- Injected velocity vector: $v = 370$ Km/s, $\delta_v = 48°$, $\alpha_v = 264°$

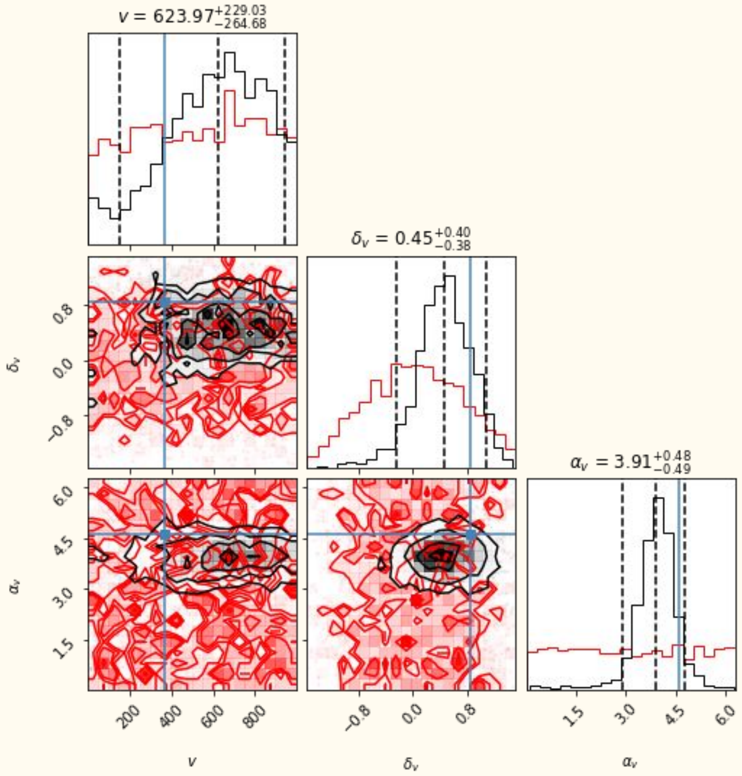
(Values inferred from CMB measurements, Planck Collaboration 2013)

Corner Plots

Red lines: priors
Black lines: posteriors



100,000 events



1,000,000 events

True value for the hyperparameters

$$v = 370.000 Km/s$$

$$\frac{v}{c} = 0.001$$

$$\delta_v = 0.838$$

$$\alpha_v = 4.608$$

Estimated value for the hyperparameters

$$v = 448.446^{+485.311}_{-405.450} Km/s$$

$$\frac{v}{c} = 0.001^{+0.001}_{-0.001}$$

$$\delta_v = 0.026^{+0.724}_{-0.703}$$

$$\alpha_v = 3.222^{+1.558}_{-1.679}$$

100,000 events

Estimated value for the hyperparameters

$$v = 623.970^{+320.266}_{-472.548} Km/s$$

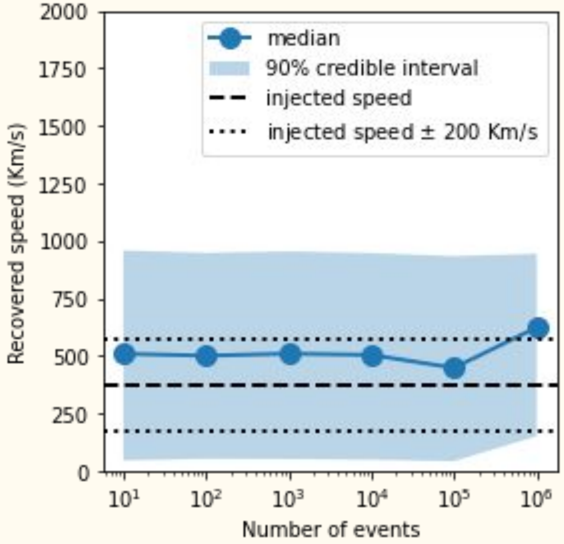
$$\frac{v}{c} = 0.002^{+0.001}_{-0.001}$$

$$\delta_v = 0.455^{+0.396}_{-0.382}$$

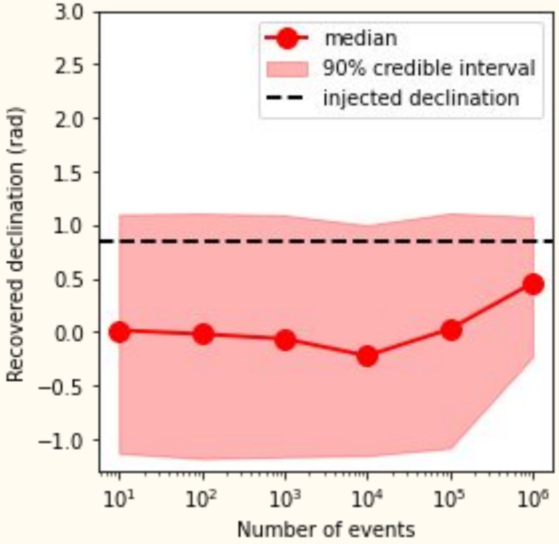
$$\alpha_v = 3.909^{+0.479}_{-0.491}$$

1,000,000 events

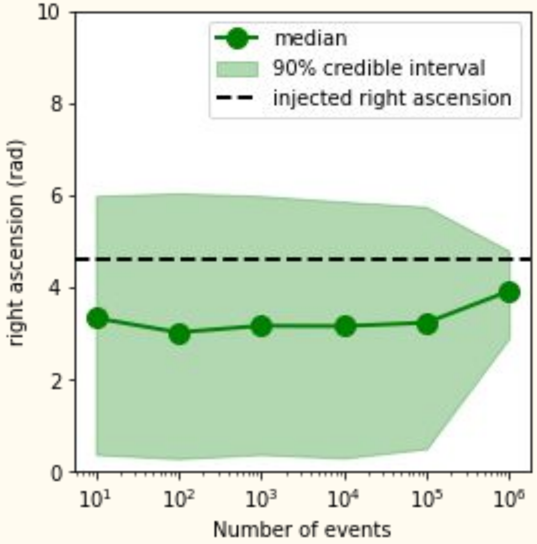
Recovery of velocity as a function of number of simulated events



Speed Measurement



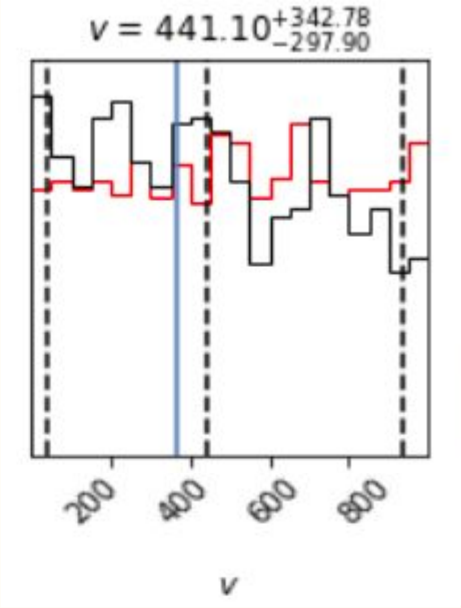
Dec Measurement



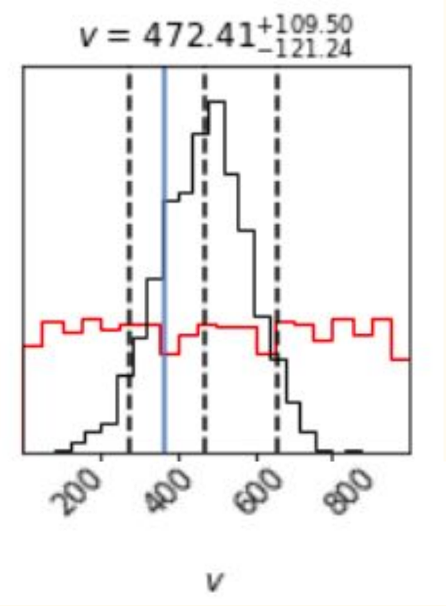
RA Measurement

Even 1 million events are not enough! How can we improve on these results?

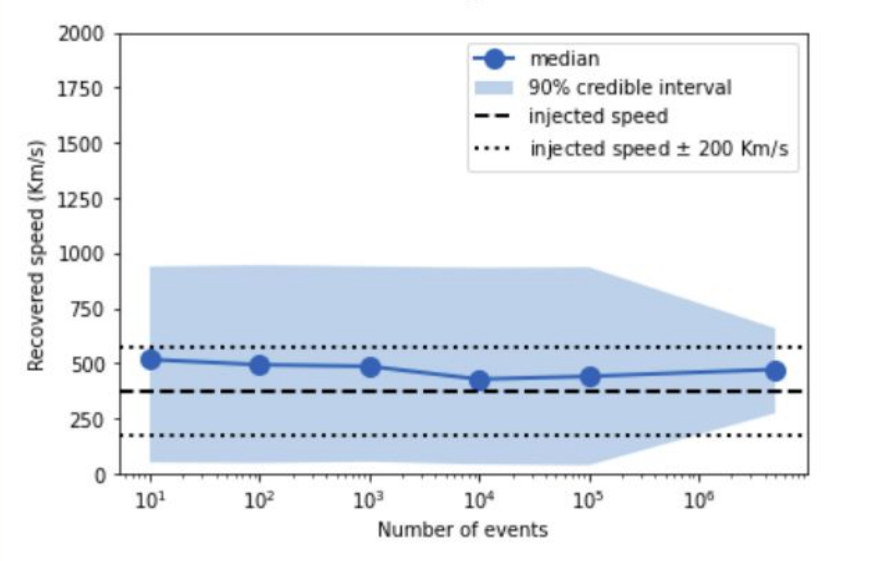
Partial Analysis with delta-function priors on direction



100,000 events



5,000,000 events



Recovery of speed as a function of number of simulated events

Can we somehow use the information contained in the mass distribution of GW sources?

Numerical Experiments-II

Hyper-model for mass and sky distribution

Event parameters : $\theta \equiv \{\delta, \alpha, m\}$
Hyper-parameters : $\Lambda \equiv \{v, \delta_v, \alpha_v, m_0, \sigma\}$

$$\pi(\delta, \alpha, m|\Lambda) \propto \frac{d^3N}{d\delta d\alpha dm}$$

Normalised PDF:

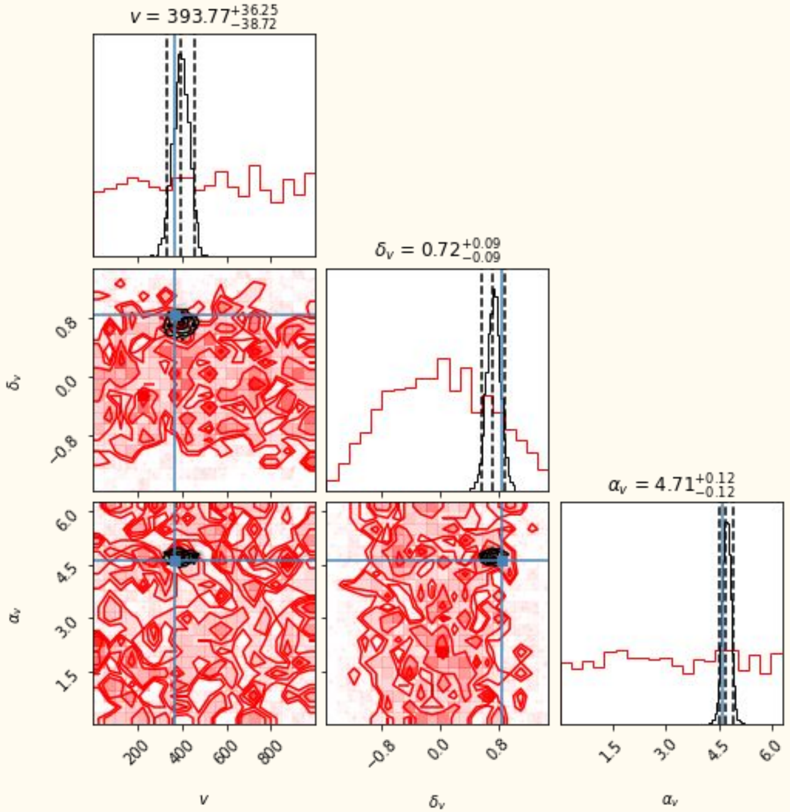
$$\pi(\delta, \alpha, m|\Lambda) = \left(\frac{\cos\delta}{4\pi}\Delta^2\right)\left(\frac{1}{\sqrt{2\pi(\sigma/\Delta)^2}}e^{-\left[\frac{(m-m_0/\Delta)^2}{2(\sigma/\Delta)^2}\right]}\right)$$

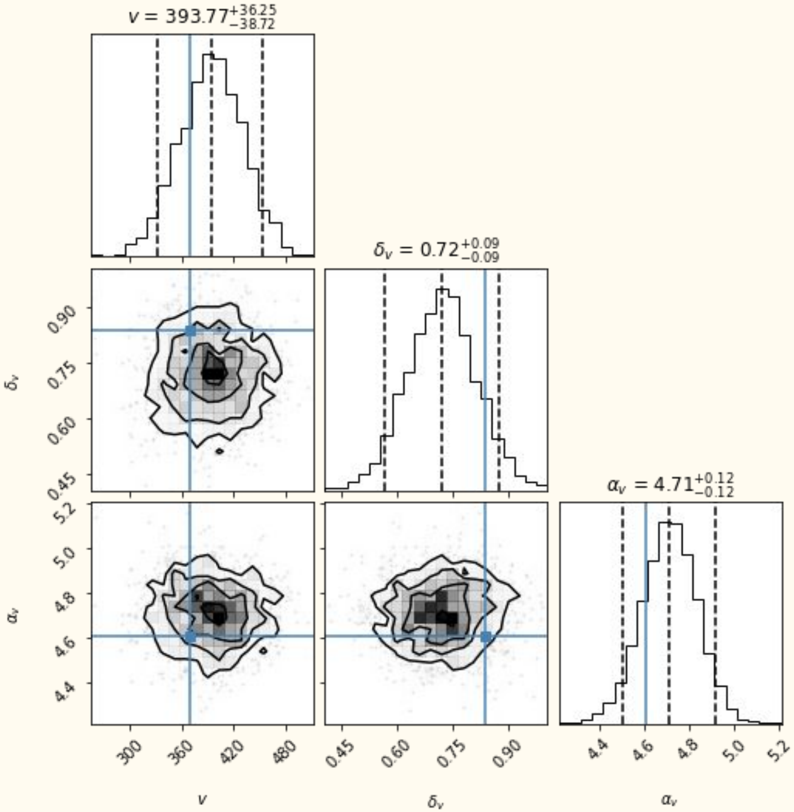
Assumptions

- Perfect point-estimates for mass and sky location of events
- There are no selection effects, ie., the 'detectors' are equally sensitive to all sky locations and all masses at all times
- The mass distribution of sources in the CMB rest frame is a gaussian
- The true mean and standard deviation in CMB rest frame mass distribution are known
- Values of the injected parameters: (Planck 2013, Farrow et al 2019)

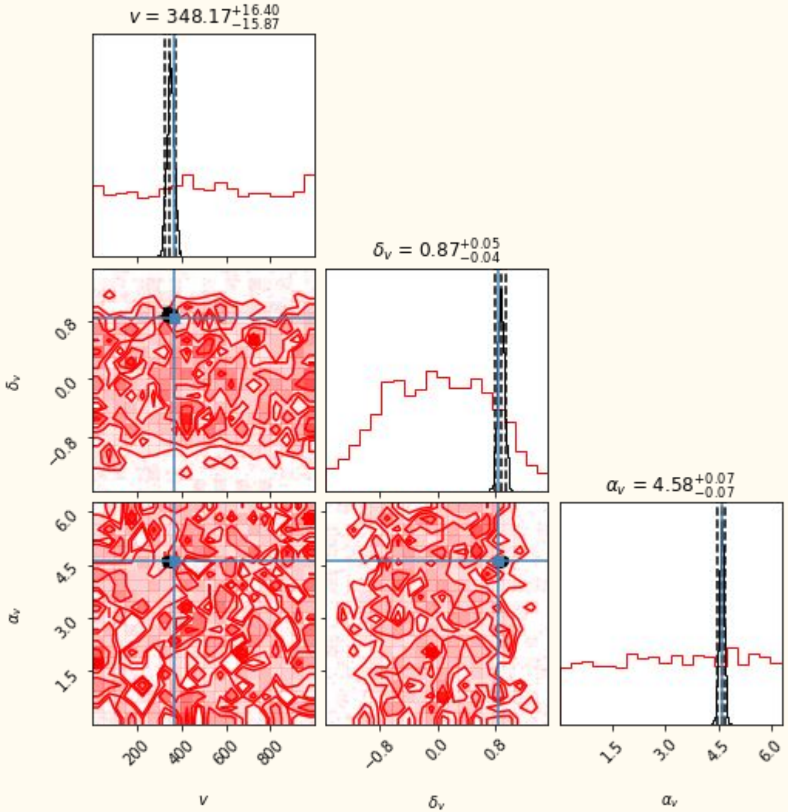
$$v = 370 \text{ Km/s},\ \delta_v = 48^\circ,\ \alpha_v = 264^\circ,\ m_0 = 1.4\ M_\odot,\ \sigma = 0.1\ M_\odot$$

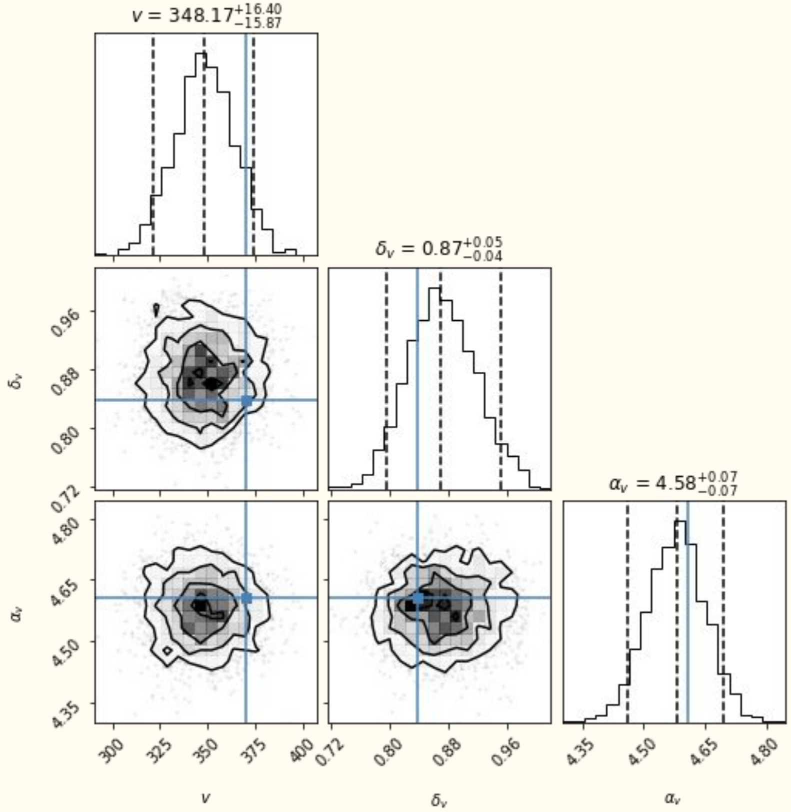
1 million events



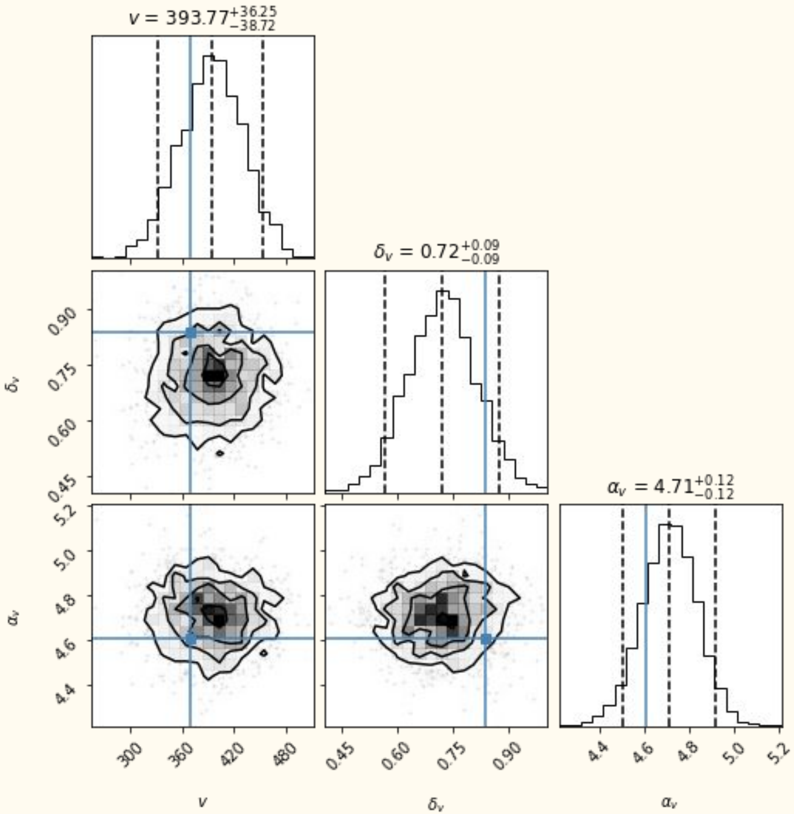


5 million events

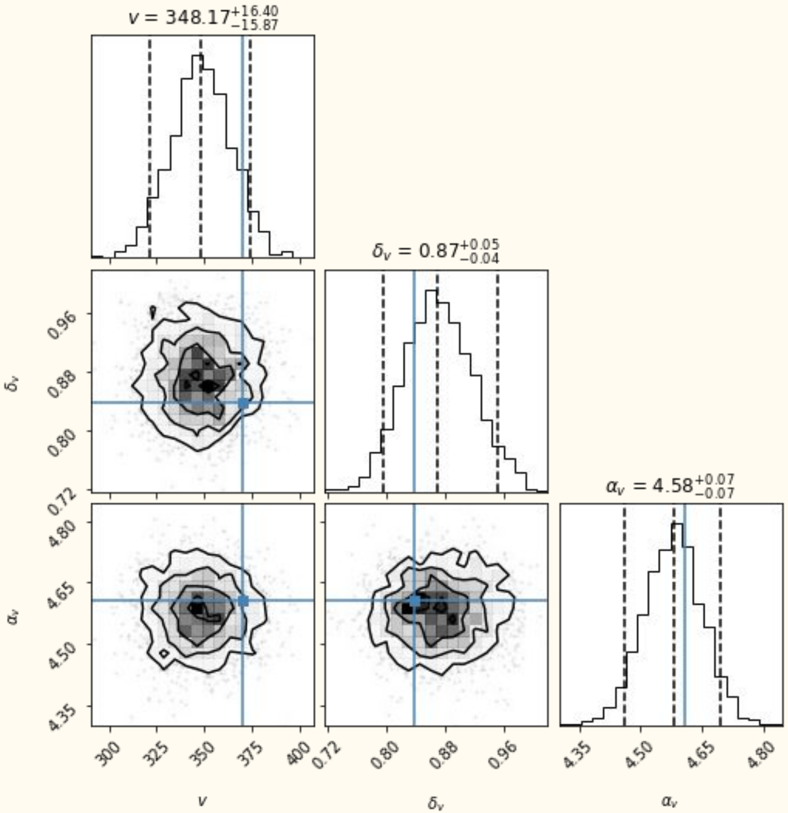




Corner Plots



1 million events



5 million events

Partial Analysis

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Estimated value for the hyperparameters
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$$v = 393.767^{+60.547}_{-62.270} Km/s$$

$$\frac{v}{c} = 0.001^{+0.000}_{-0.000}$$

$$\delta_v = 0.721^{+0.093}_{-0.093}$$

$$\alpha_v = 4.709^{+0.123}_{-0.125}$$

1 million events

```
Estimated value for the hyperparameters
```

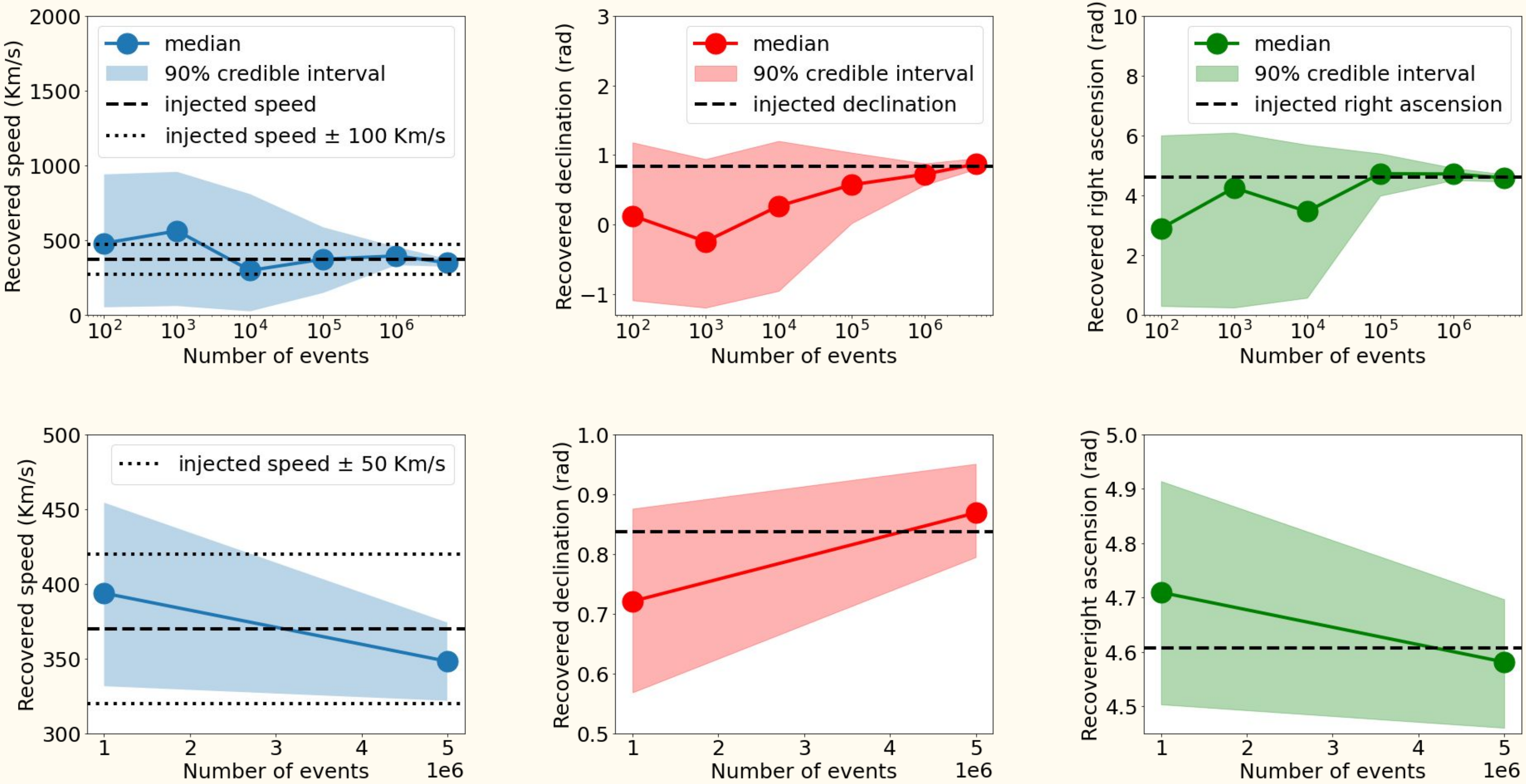
$$v = 348.171^{+25.974}_{-26.403} Km/s$$

$$\frac{v}{c} = 0.001^{+0.000}_{-0.000}$$

$$\delta_v = 0.869^{+0.050}_{-0.045}$$

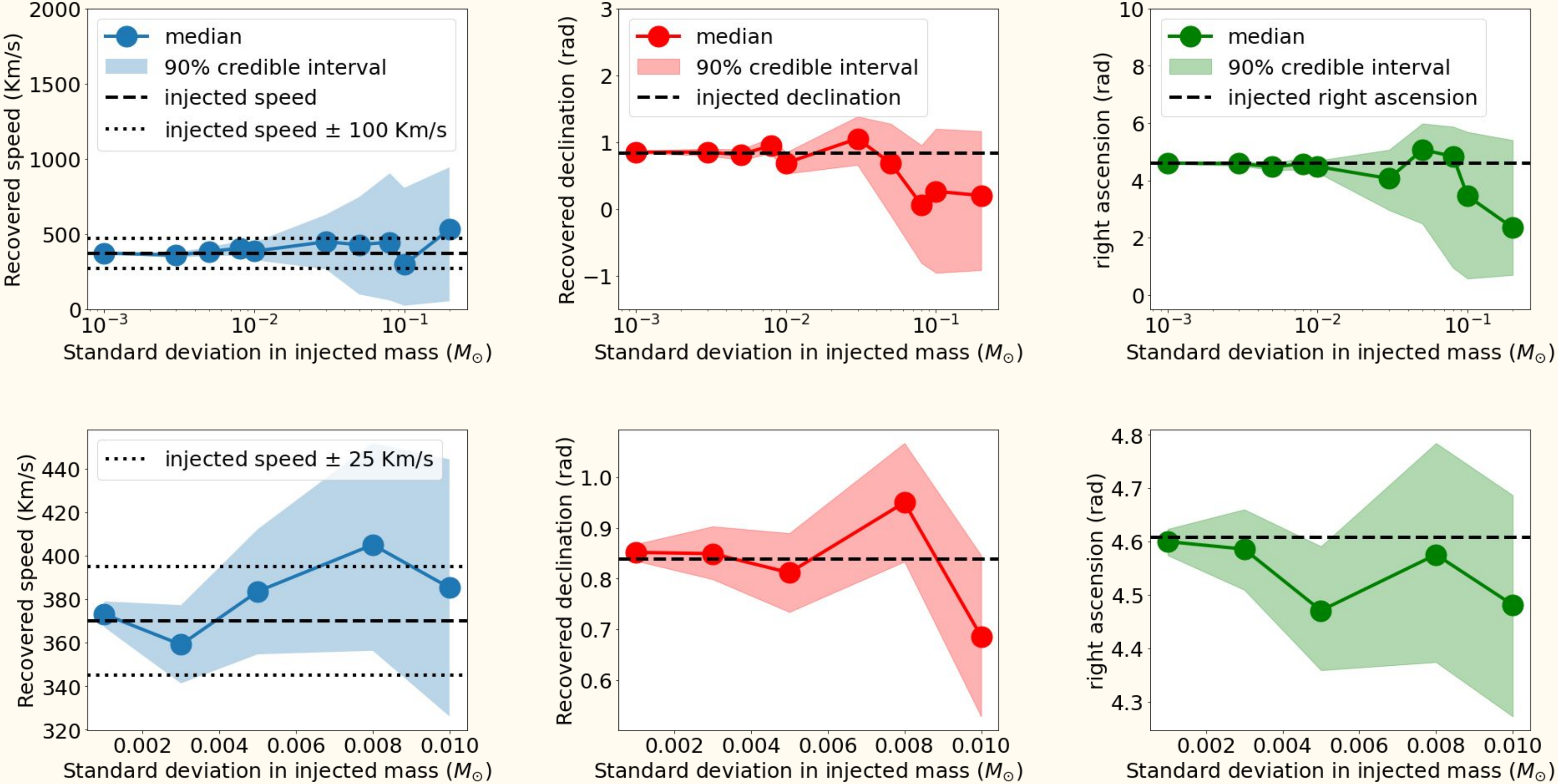
$$\alpha_v = 4.581^{+0.070}_{-0.075}$$

5 million events



Recovery of velocity as a function of number of simulated events

What happens if we vary the standard deviation in the mass distribution, keeping the number of events fixed?

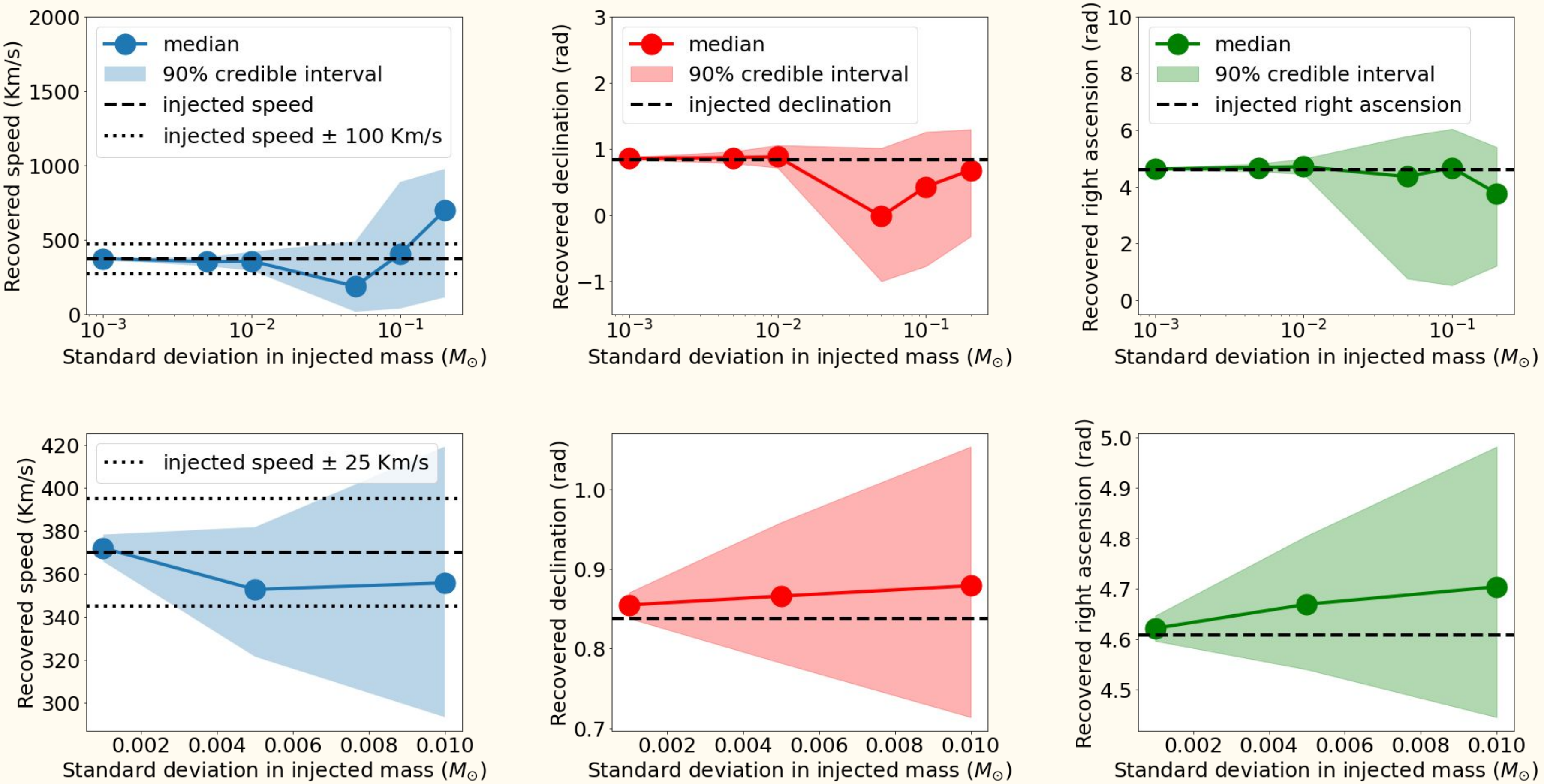


Recovery of velocity as a function of standard deviation for 10k events

Why does a narrower distribution in mass allow for better measurements?

What if we relax one of the assumptions?

- Uncertainty in the localisation of sources: 5 square degrees
- Posterior distribution of sky coordinates for each event: 2-D gaussian
- There are still no selection effects, ie., the ‘detectors’ are equally sensitive to all sky locations at all times



Recovery of velocity as a function of standard deviation for 10k events

A Note on Assumptions

- We assumed perfect point estimates for both mass and sky location of all the events. Is this reasonable?
- 3G detectors are predicted to be able to localise the sources to 1 square degree (Hall and Evans 2019) in the sky
- The assumption of perfect estimate for mass needs to be relaxed in future work
- We assumed that there are no selection effects i.e., we will observe all events that happen. How realistic is this assumption?
- 3G detectors are predicted to have no selection effects
- We assume that the rest-frame mass distribution is a Gaussian
- This is valid for galactic double neutron stars (Farrow, Zhu, and Thrane 2019)
- However, Not all galactic BNS would merge in the hubble time (a crucial requirement to observe them in gravitational-waves)

Conclusions

- We discussed the framework of hierarchical Bayesian inference framework, and motivated how we can use it to measure dipole anisotropy in the distribution of gravitational-wave sources and constrain Earth's motion
- We discussed a budding cosmological tension, and how we can potentially use a population of merging compact binaries to resolve it
- Using only sky locations of sources, we were not able to confidently recover the injected dipole even with 5 million perfectly localised events
- Assuming a narrow gaussian mass function, we were able to obtain reasonable constraints on motion of earth with around 1 million events
- We discussed how a narrower distribution in mass allows for better constraints on the speed of earth

Future Directions

- Perform a full 5-dimensional inference including the population mean mass and standard deviation as hyperparameters
- Incorporate other features in the mass distribution such as edges in the analysis
- Account for selection effects
- Relax the assumption of knowing perfect point estimates for all parameters of all sources
- Work with more realistic population mass distributions

References

1. *Planck 2013 results. XXVII. Doppler boosting of the CMB: Eppur si muove* Planck collaboration, Astron.Astrophys. 571 (2014) A27
2. *A Test of the Cosmological Principle with Quasars*, Nathan J. Secrest et al 2021 ApJL 908 L51
3. *Dipole Anisotropy in Gravitational Wave Source Distribution*, Kashyap et al 2022
4. *An introduction to Bayesian inference in gravitational-wave astronomy: parameter estimation, model selection, and hierarchical models*, Eric Thrane, Colm Talbot Publ.Astron.Soc.Austral. 36 (2019) e010
5. *The population of merging compact binaries inferred using gravitational waves through GWTC-3*, The LIGO Scientific Collaboration 2021
6. *Metrics for next-generation gravitational-wave detectors*, Evan D. Hall and Matthew Evans
7. *The Mass Distribution of Galactic Double Neutron Stars*, Nicholas Farrow, Xing-Jiang Zhu and Eric Thranel, The Astrophysical Journal, 876:18, 2019 May 1

Thank you!

Questions?

Supplemental Slides

Detection Rate of BNS

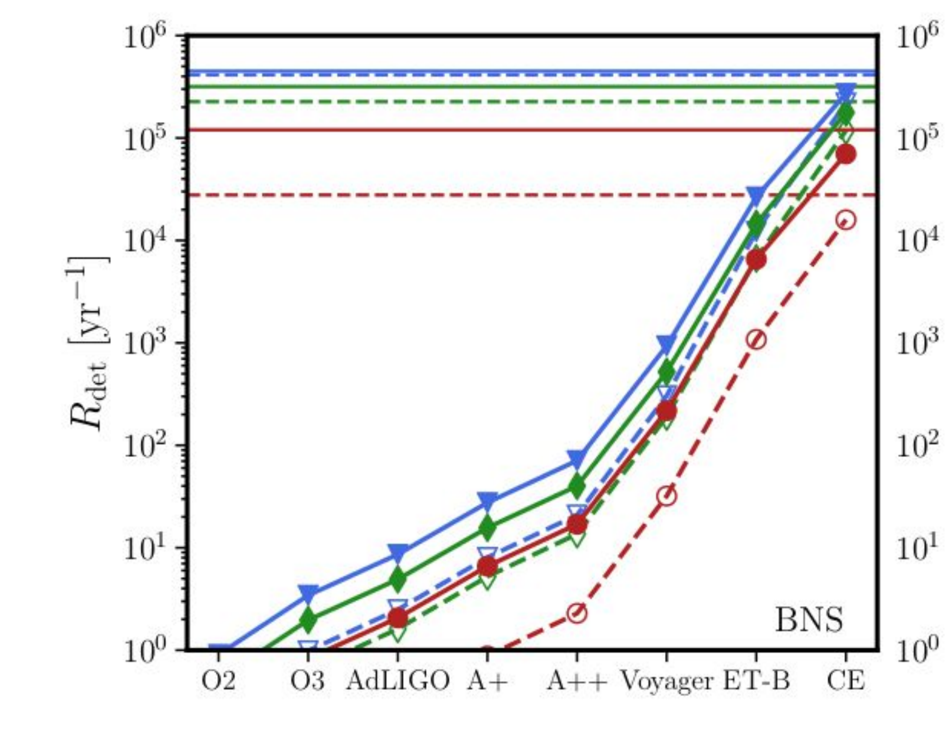


Image Credit: Baibhav et. al 2019

Detection rates of BNSs for second- and third-generation detectors

$$R = \int_0^{z_{max}} R(z) \frac{dV_c}{dz} \frac{1}{1+z} dz$$

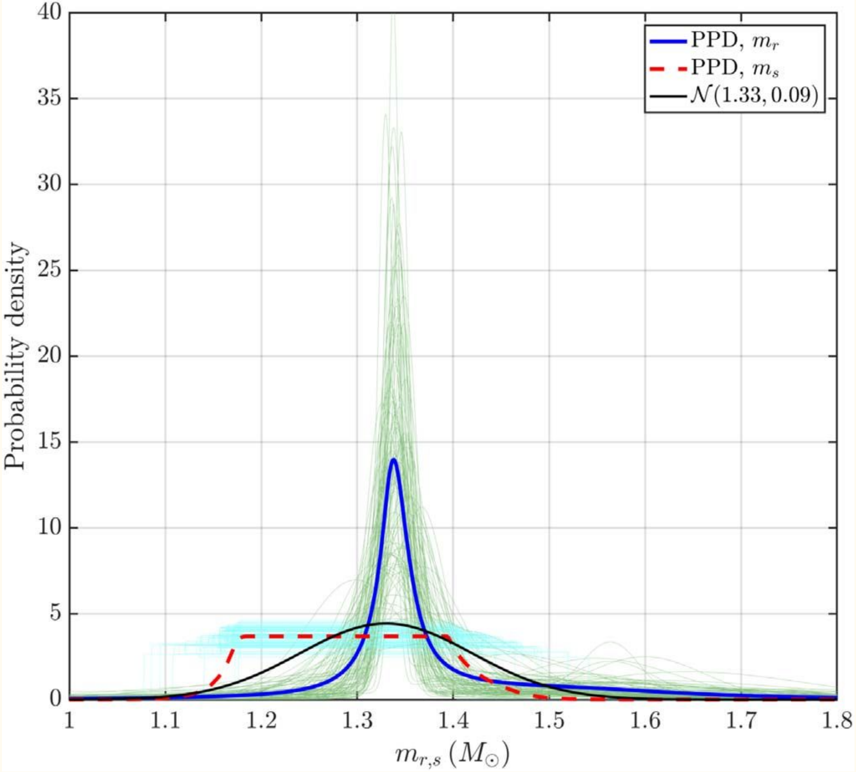
$$\approx R_0 \int_0^{z_{max}} \frac{dV_c}{dz} \frac{1}{1+z} dz$$

$$R(z) = \frac{dN}{dV_c dt}$$

$$R_0 \sim 300\ \mathrm{Gpc}^{-3}\ \mathrm{yr}^{-1}; z_{max} \sim 5$$

$$\implies R \sim 10^5\ \mathrm{yr}^{-1}$$

Mass distribution of Galactic Double Neutron Stars



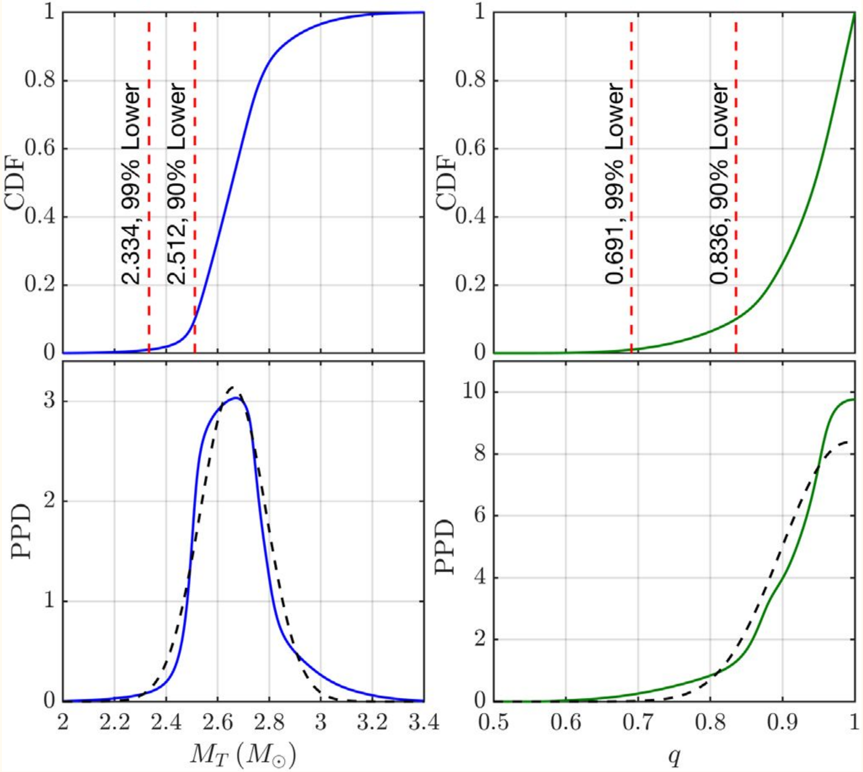


Image Credit: Nicholas Farrow, Xing-Jiang Zhu and Eric Thrane