Berezinskii-Kosterlitz-Thouless Transition:

Vortices, Topology, and all that jazz

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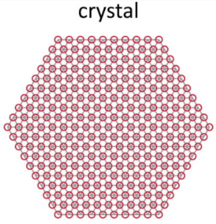
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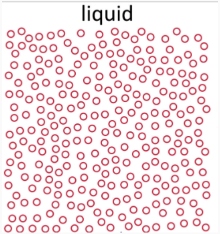
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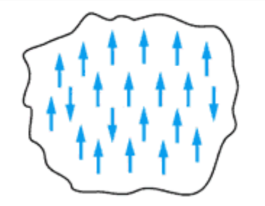
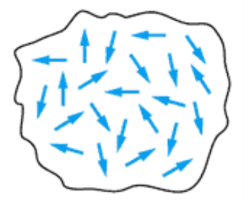
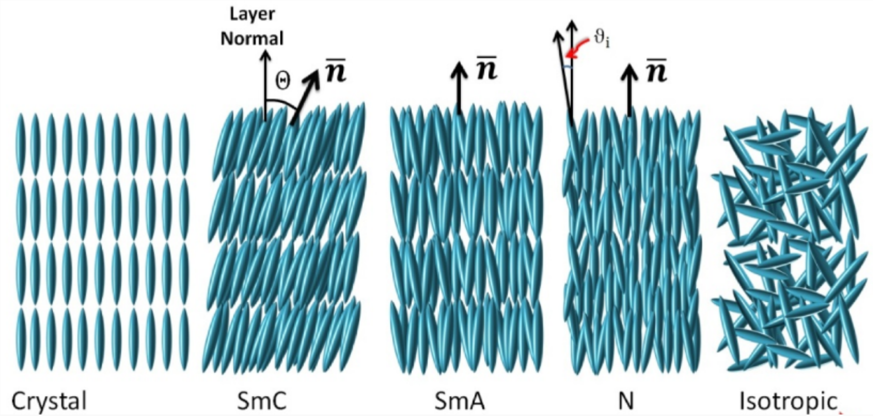


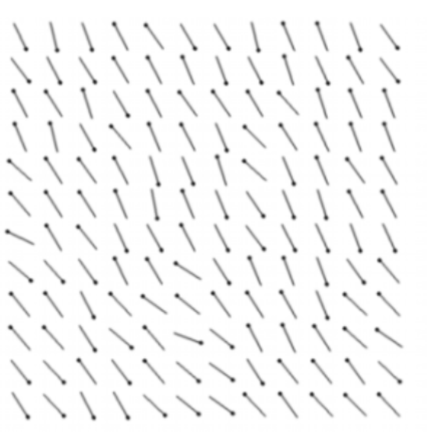


Phases and Phase Transitions

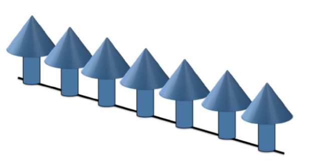


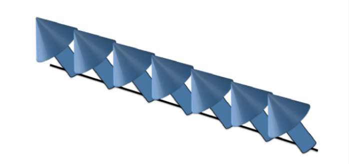
The XY Model

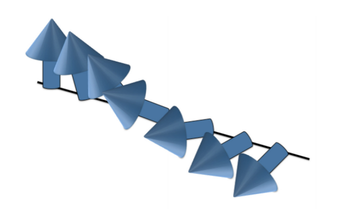
$$H = -J \sum_{\langle i,j \rangle} \vec{S_i} \cdot \vec{S_j} - \vec{B} \cdot \sum_i \vec{S_i}$$



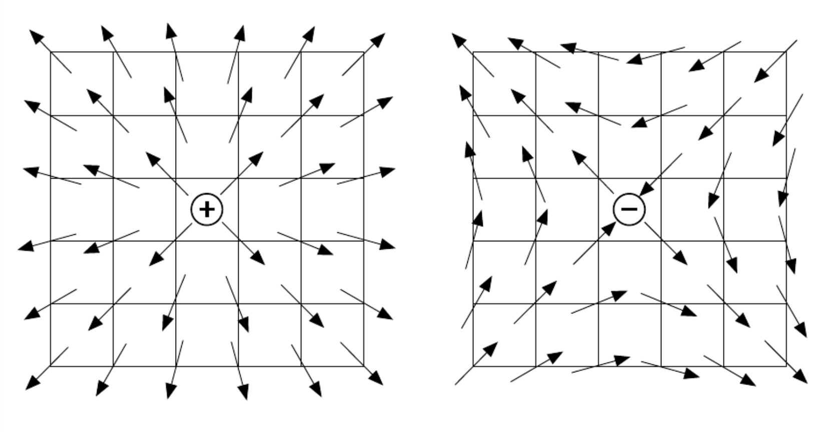
Spin Waves



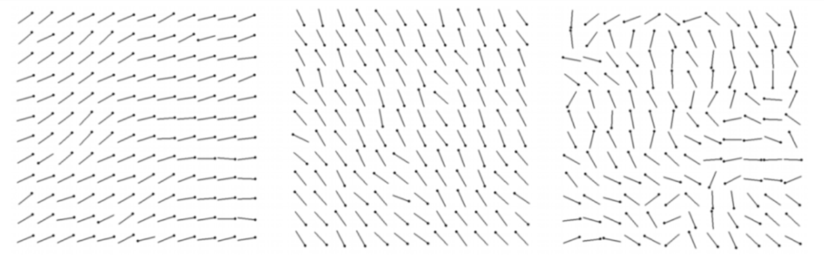




Vortices!



The Phase Transition



Stat Mech: Interested in studying the property of many-body systems.
We are interested in macroscopic behaviour.

Interacting systems show interesting property such as the existence of different phases.

Phases and Phase Transition

Phase itself not well defined.
Wherever a particular equation of state holds.
We ask a set of Yes/No questions. Depending on answer we define phases.
Use of order-parameters.
Give examples of different phases

Phase transition → Qualitative change in the behaviour of the system
Characterized using thermodynamic quantities.
F is non-analytic.
↳ Some derivative of F blows up. → Due to Ehrenfest

→ involves latent heat

First order: $V = -\frac{\partial F}{\partial p}$, $m = \frac{\partial F}{\partial B}$, $S = -\frac{\partial F}{\partial T}$

Second order:
↓
Continuous

$$\chi = \frac{\partial M}{\partial B} = \frac{\partial^2 F}{\partial B^2}; \quad C_V = -T\frac{\partial^2 F}{\partial T^2}$$

↳ ∞ correlation length

Fluctuation - dissipation: $\chi = \frac{1}{kT}\sum_n \Gamma(n)$ $\quad \Gamma(n) = \langle s(n)s(0)\rangle$

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How we characterize different phases.
Use order-parameter } Due to Landau
Broken symmetries

How to detect phase transitions?
Diverging quantities.
Change in correlation functions, etc.

Comment on the point of models in stat physics.

XY Model: → Magnet, superfluid, Josephson junc. arrays

dof: 2D unit vectors. Lattice: Square 2D.

$$H = -J \sum_{\langle ij \rangle} \bar{S}_i \cdot \bar{S}_j + \bar{B} \cdot \sum_i \bar{S}_i$$

$$= -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

We can think of it as a magnet model. But connections to superfluids

Mermin-Wagner Theorem: Long enough spins are uncorrelated.
No order.

→ Give spin-wave images.

But computational and other evidence for phase transition.

$$Z = \int \frac{d\theta_i}{2\pi} e^{+\beta J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)}$$

$$= \int \frac{d\theta_i}{2\pi} \; e^{K \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)}$$

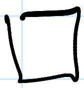
$$e^{-\beta H} = \prod_{\langle ij \rangle} \left[1 + K\cos(\theta_i - \theta_j) \right]$$

$$= 1 + \sum_{\langle ij \rangle} K \cos(\theta_i - \theta_j)$$

$$+ \sum_{\substack{\langle ij \rangle \\ \langle lm \rangle}} K^2 \cos(\theta_i - \theta_j) \cos(\theta_l - \theta_m)$$

Bond picture $\int \frac{d\theta}{(2\pi)} \cos(\theta - \theta_1) \cos(\theta - \theta_2)$

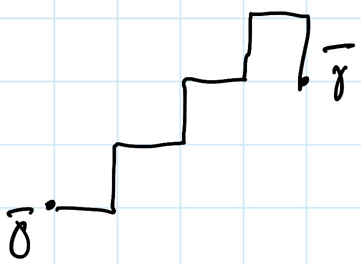
$$= \frac{1}{2} \cos(\theta_1 - \theta_2)$$

First non-zero contribution:  $\rightarrow$ order $K^4 \cos(\theta_1 - \theta_2) \times \cos(\theta_2 - \theta_3)$
$\cos(\theta_3 - \theta_4) \cos(\theta_4 - \theta_1)$

Sum. over closed loops.

Similarly $\langle \vec{S}_0 \cdot \vec{S}_{\vec{r}} \rangle =$

$\left(\frac{K}{2}\right)^{\text{No. of bonds.}}$



$$\langle \vec{S}_0 \cdot \vec{S}_{\vec{r}} \rangle \sim \left(\frac{K}{2}\right)^{r} = e^{-r/\xi} \qquad \xi = \frac{1}{\ln(2/K)}$$

Low Temp. Expansion:

Continuum theory → $H = \frac{1}{2}\int d^2r\, J\,(\nabla\varphi)^2$

↓

Comes from expanding $\cos(\theta_i - \theta_j)$

This gives us:

$$\frac{1}{2}\langle(\theta_0 - \theta_{\bar{r}})^2\rangle = \frac{1}{2\pi K}\ln\left(\frac{r}{a}\right)$$

$$\langle \bar{S}_0 \cdot \bar{S}_r\rangle \approx \left(\frac{a}{r}\right)^{\frac{1}{2\pi K}}$$

→ True for any generic continuous spin system

There is a blow up in $\langle S_0 \cdot S_r\rangle$ as K goes from ∞ to 0

This is just an indication, not a proof

What is this? Let's look at excitations in X-Y model.

Topological Charge

Energy of a vortex: $\bar{\nabla}\theta = -n\bar{\nabla}\times(\hat{z}\ln r)$

$$\oint \bar{\nabla}\theta \cdot d\bar{s} = \frac{d\theta}{dr}(2\pi r) = 2\pi n \Rightarrow \frac{d\theta}{dr} = \frac{n}{r}$$

$$\beta\varepsilon_n = \beta\varepsilon_n^0(a) + \frac{K}{2}\int_a d^2r\,(\bar{\nabla}\theta)^2$$

→ Large cost

$$= \beta\varepsilon_n^0(a) + \pi K n^2 \ln\left(\frac{L}{\ }\right)$$

L → IR cutoff