

What is Chaos?

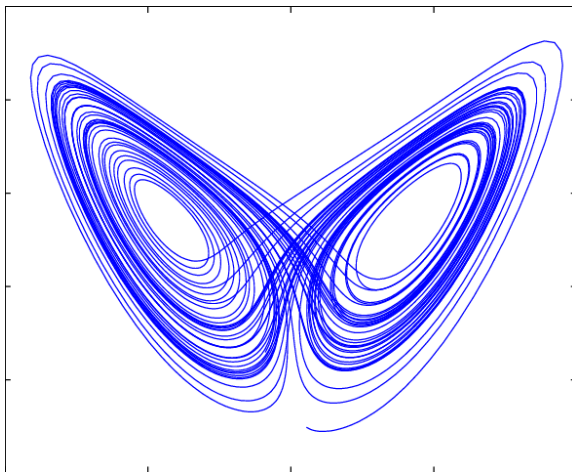


Figure: Lorenz Attractor

What is Chaos?

A deterministic dynamical system is said to exhibit chaotic behaviour when its dynamics are highly sensitive to the initial conditions and can become effectively unpredictable over long times.

Features

- ▶ **Deterministic dynamics:** The future evolution is completely determined by the equations of motion and the initial conditions.
- ▶ **Sensitive dependence on initial conditions:** Two trajectories starting arbitrarily close to each other can separate exponentially with time.
- ▶ **Aperiodic dynamics:** The motion does not settle into a simple periodic orbit.
- ▶ **Bounded motion:** In many physical systems, chaotic trajectories remain confined to a finite region of phase space.

Sensitive Dependence on Initial Conditions

Consider two trajectories with initially small separation

$$\delta(0) = |\mathbf{x}_1(0) - \mathbf{x}_2(0)|$$

For a chaotic system, this separation typically grows approximately as

$$\delta(t) \sim \delta(0)e^{\lambda t}$$

where λ is the **Lyapunov exponent**.

The largest Lyapunov exponent is defined as

$$\lambda_{\max} = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \left(\frac{\delta(t)}{\delta(0)} \right)$$

Chaos does **not** mean randomness. The equations of motion remain deterministic; the apparent randomness arises from the exponential amplification of small uncertainties in the initial conditions.

Quantum Chaos?

If a classical system is chaotic, what does chaos mean in the corresponding quantum system?

Quantum mechanics introduces a fundamental difficulty:

- ▶ Quantum states evolve according to the **Schrödinger equation**, which is linear and deterministic: $(i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H|\psi(t)\rangle)$
- ▶ The evolution of a closed quantum system is **unitary** ($|\psi(t)\rangle = U(t)|\psi(0)\rangle$, $U(t) = e^{-iHt/\hbar}$).
- ▶ Two initially close quantum states do not separate exponentially in the same way as classical trajectories.
- ▶ The uncertainty principle prevents us from defining a quantum trajectory in phase space with simultaneously precise position q and momentum p .

The classical definition of chaos based on sensitive dependence on initial conditions does not have a direct quantum analogue.

Then What is Quantum Chaos?

Although the classical definition of chaos does not directly apply to quantum mechanics, quantum systems can still display signatures of classical chaotic dynamics.

Instead of following individual trajectories, one studies properties such as

- ▶ Energy-level statistics
- ▶ Eigenstate statistics
- ▶ Spectral correlations
- ▶ Dynamical correlations

Random Matrix Theory

Random Matrix Theory was originally developed in nuclear physics to describe complicated spectra statistically.

A key question was whether the spectral statistics of a quantum system are related to the nature of its classical dynamics.

Berry–Tabor, 1977

For generic quantum systems with an integrable classical limit, the unfolded energy levels exhibit approximately Poisson statistics:

$$P(s) = e^{-s}.$$

What happens when the classical dynamics is chaotic?

Random Matrix Theory

Bohigas, Giannoni and Schmit studied the quantum Sinai billiard, whose classical dynamics is chaotic. They found that its spectral fluctuations were consistent with the **Gaussian Orthogonal Ensemble (GOE)** of Random Matrix Theory.

BGS, 1984

It states that the energy level fluctuations of a quantum system whose classical limit is chaotic share the same statistical properties as the appropriate Random Matrix Theory ensemble determined by the system's fundamental symmetries.

Random Matrix Theory

Random Matrix Theory (RMT) provides a statistical framework for describing the spectral properties of complex quantum systems whose microscopic details are unknown or too complicated to analyze directly.

Instead of studying a specific Hamiltonian, RMT considers an ensemble of random matrices satisfying the relevant physical symmetries.

Many spectral properties do not depend on the microscopic details of the system, but are determined by its underlying symmetry class.

Which spectral properties are universal for a given symmetry class?

Gaussian Random Matrix Ensembles

The three classical Gaussian ensembles are classified by the Dyson index β .

Ensemble	β	Entries	Matrix Symmetry
GOE	1	Real	Symmetric
GUE	2	Complex	Hermitian
GSE	4	Quaternion	Self-dual

- ▶ **GOE:** Time-reversal symmetric systems with $T^2 = +1$.
- ▶ **GUE:** Systems without time-reversal symmetry.
- ▶ **GSE:** Time-reversal symmetric systems with $T^2 = -1$, typically associated with half-integer spin.

Joint Probability Distribution

Let the eigenvalues of an $N \times N$ random matrix be $\lambda_1, \lambda_2, \dots, \lambda_N$. For the Gaussian ensembles, their joint probability density is

$$P(\{\lambda_i\}) \propto e^{-\frac{1}{2} \sum_i \lambda_i^2} \prod_{i < j} |\lambda_i - \lambda_j|^\beta$$

The first factor determines the overall extent of the spectrum, while the Vandermonde factor

$$\prod_{i < j} |\lambda_i - \lambda_j|^\beta$$

determines the repulsion between eigenvalues.

The eigenvalues of Gaussian random matrices tend to avoid degeneracies. The strength of this repulsion depends on β .

Wigner Semicircle Law

The individual eigenvalues fluctuate from one random matrix realization to another, but their overall density becomes universal for large matrix dimension.

For the standard normalization,

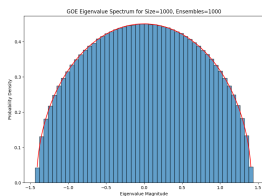
$$\boxed{\rho(\lambda) = \frac{1}{\pi} \sqrt{2 - \lambda^2}} \quad |\lambda| \leq \sqrt{2}.$$

This is the **Wigner semicircle law**. In the large- N limit,

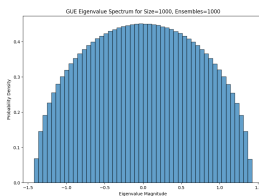
$$\rho_N(\lambda) \longrightarrow \rho(\lambda)$$

The semicircle describes the global density of eigenvalues. It does not contain information about the detailed correlations between individual levels.

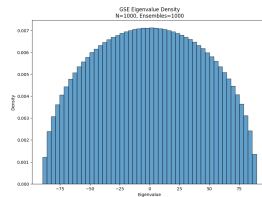
Gaussian Ensembles



(a) GOE



(b) GUE



(c) GSE

Figure: Eigenvalue Distribution of Gaussian Ensembles

Finite- N Effects

For a finite matrix dimension N , the numerical eigenvalue density fluctuates around the Wigner semicircle.

- ▶ Small N produces visible statistical fluctuations.
- ▶ Increasing N makes the histogram progressively smoother.
- ▶ In the large- N limit, the numerical density approaches the Wigner semicircle.

$$\rho_N(\lambda) \rightarrow \rho_{\text{Wigner}}(\lambda) \quad (N \rightarrow \infty)$$

Finite-N Gaussian Ensembles

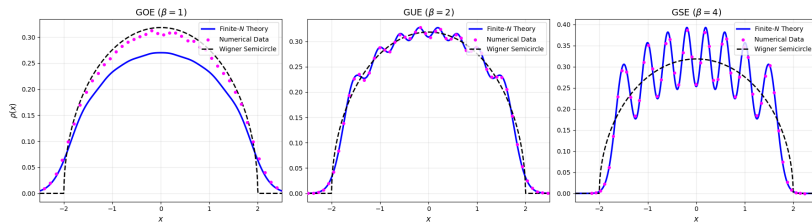


Figure: Finite-N Eigenvalue Distribution

Energy-Level Spacing

Consider the ordered eigenvalues of a Hamiltonian

$E_1 < E_2 < \dots < E_N$. The nearest-neighbour spacing is defined as $s_i = E_{i+1} - E_i$.

The collection $\{s_1, s_2, \dots, s_{N-1}\}$ contains information about the correlations between neighbouring energy levels.

For an uncorrelated spectrum, neighbouring levels behave approximately independently.

For a chaotic quantum system, neighbouring levels exhibit strong correlations due to level repulsion.

Unfolding the Spectrum

The mean density of states generally varies across the spectrum:

$$\rho(E) = \frac{dN(E)}{dE}.$$

Therefore, the raw spacings $s_i = E_{i+1} - E_i$ do not have a uniform mean across the spectrum.

We therefore **unfold** the spectrum by replacing each eigenvalue E_i with

$$\varepsilon_i = \overline{N}(E_i),$$

where $\overline{N}(E)$ is the smooth part of the cumulative level-counting function. The unfolded spacing is then

$$s_i = \varepsilon_{i+1} - \varepsilon_i.$$

By construction, $\langle s \rangle = 1$.

Nearest-Neighbour Spacing Distribution

After unfolding, we study the probability distribution $P(s)$ of the nearest-neighbour spacings. It is normalized as $\int_0^\infty P(s) ds = 1$, with $\int_0^\infty sP(s) ds = 1$

The behaviour of $P(s)$ near $s = 0$ is particularly important.

Level repulsion

For chaotic quantum systems, nearby energy levels tend to repel each other:

$$P(s) \rightarrow 0 \quad \text{as} \quad s \rightarrow 0$$

For uncorrelated levels, there is no such level repulsion.

Poisson Statistics

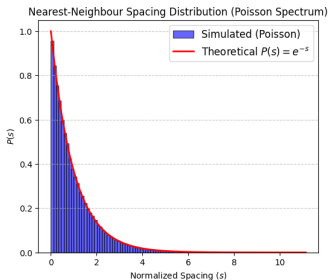


Figure: Poisson spacing distribution

For an uncorrelated spectrum, the nearest-neighbour spacings follow the Poisson distribution

$$P_P(s) = e^{-s}$$

At small spacing, $P_P(0) = 1$. Therefore, there is **no level repulsion**.

This behaviour is typically associated with quantum systems whose classical counterparts are integrable, under the appropriate assumptions.

Wigner-Dyson Statistics

For chaotic quantum systems, Random Matrix Theory predicts level repulsion. For the Gaussian ensembles, the Wigner surmise is

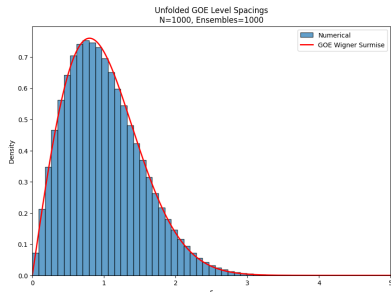
$$P_{\beta}(s) = a_{\beta} s^{\beta} e^{-b_{\beta} s^2}$$

where β specifies the symmetry class.

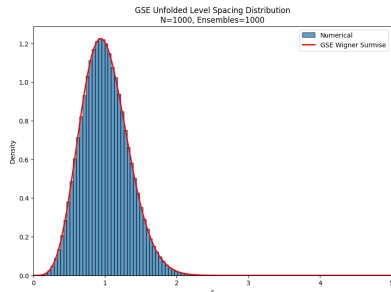
For the three classical Gaussian ensembles,

Ensemble	β	$P_{\beta}(s)$
GOE	1	$\frac{\pi}{2} s e^{-\pi s^2/4}$
GUE	2	$\frac{32}{\pi^2} s^2 e^{-4s^2/\pi}$
GSE	4	$\frac{2^{18}}{3^6 \pi^3} s^4 e^{-64s^2/(9\pi)}$

Wigner-Dyson Statistics



(a) GOE Unfolded Spacing Distribution



(b) GSE Unfolded Spacing Distribution

Figure: Eigenvalue Spacing Distribution after Unfolding

Spacing Ratio Statistics

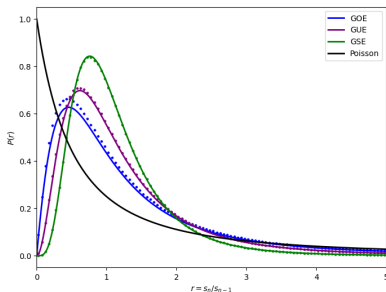
Instead of considering the spacing itself, we compare two consecutive spacings:

$$r_i = \frac{\min(s_i, s_{i+1})}{\max(s_i, s_{i+1})}$$

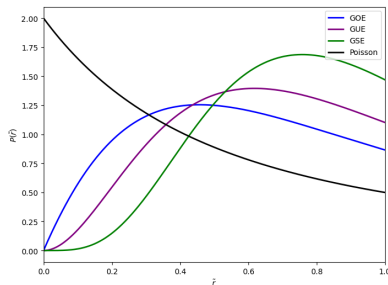
The ratio compares the relative sizes of neighbouring gaps and does not require the spectrum to be unfolded.

Statistics	Poisson	GOE	GUE	GSE
$\langle r \rangle$	∞	1.750	1.361	1.1747
$\langle \tilde{r} \rangle$	0.3863	0.5307	0.5996	0.6744

Spacing Ratio Statistics



(a) Distribution of Spacing Ratios



(b) Distribution of Spacing Ratios (Folded)

Therefore, the mean spacing ratio can be used to distinguish between different spectral regimes without explicitly unfolding the spectrum.

Number Variance

Let $n(L)$ denote the number of eigenvalues contained in an interval of length L .

Number Variance

$$\Sigma^2(L) = \langle (n(L) - L)^2 \rangle$$

where the average is taken over different intervals.

For an uncorrelated Poisson spectrum,

$$\Sigma^2(L) = L$$

indicating that the fluctuations grow linearly with the interval length.

In contrast, Gaussian ensembles exhibit spectral rigidity, and the number variance increases only logarithmically:

$$\Sigma^2(L) \sim \frac{2}{\beta\pi^2} \ln L$$

Number Variance

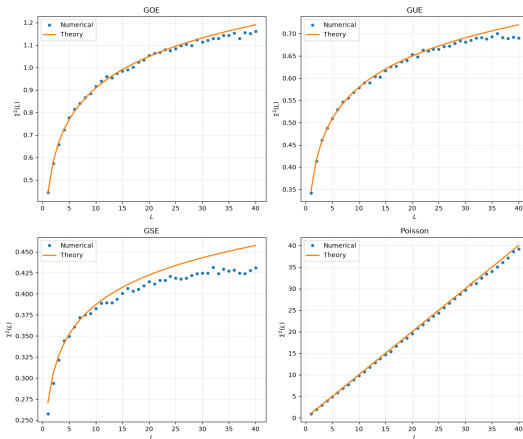


Figure: Number Variance for various types of Ensembles

Dyson-Mehta Δ_3 Statistic

$\Delta_3(L)$ Statistic

$$\Delta_3(L) = \left\langle \min_{A,B} \frac{1}{L} \int_E^{E+L} [N(E) - AE - B]^2 dE \right\rangle$$

where $N(E)$ is the cumulative counting function.

For a Poisson spectrum,

$$\Delta_3(L) = \frac{L}{15}$$

so the rigidity grows linearly with L .

For Gaussian ensembles, spectral rigidity is much stronger,

$$\Delta_3(L) \sim \frac{1}{\beta\pi^2} \ln L,$$

Thus, both $\Sigma^2(L)$ and $\Delta_3(L)$ distinguish uncorrelated spectra from correlated spectra.

Dyson Mehta Statistic

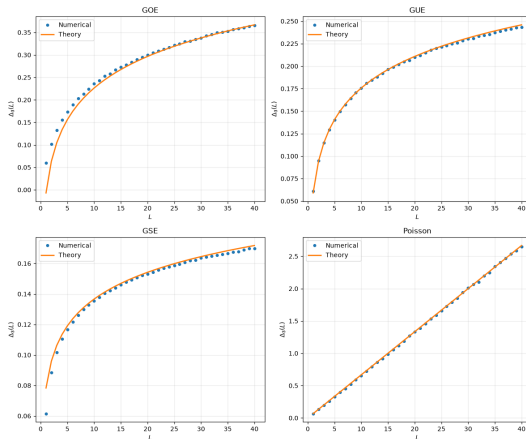


Figure: $\Delta_3(L)$ for various types of Ensembles

Classical Kicked Rotor

Hamiltonian of a Classical Kicked Rotor

$$H = \frac{p^2}{2I} + k \cos x \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

After properly scaling the equation and integrating, we get a pair of difference equations,

$$x_{n+1} = x_n + p_{n+1}, \quad p_{n+1} = p_n + K \sin(x_n)$$

Standard Map of Classical Chaos

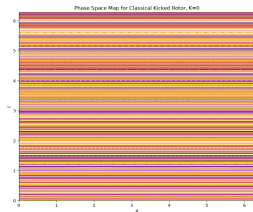


Figure: $K = 0$

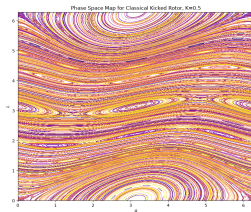


Figure: $K = 0.5$

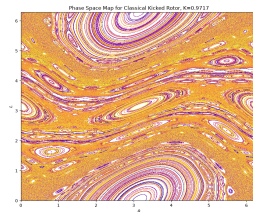


Figure: $K = 0.9717$

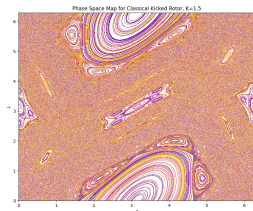


Figure: $K = 1.5$

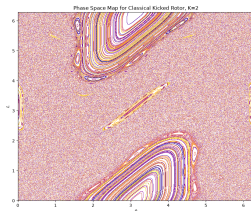


Figure: $K = 2$

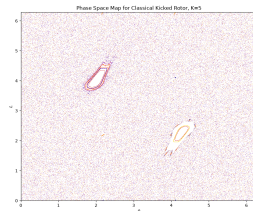


Figure: $K = 5$

Quantum Kicked Rotor

The quantum kicked rotor is described by the time-dependent Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = -\frac{\hbar^2}{2I} \frac{\partial^2}{\partial x^2} \psi(x, t) + k \cos x \sum_{n=-\infty}^{\infty} \delta(t - nT) \psi(x, t)$$

The scaled parameter $K(= \frac{kT}{I})$ acts as the classical stochasticity parameter, controlling the transition to chaos.

Floquet Operator

Since the Hamiltonian is periodic, we can use Floquet theory to describe it. Since the evolution between kicks is free and the kicks are instantaneous, the one-period Floquet operator factors as

$$\hat{U} = \hat{U}_{\text{free}} \hat{U}_{\text{kick}} = \exp\left(-i \frac{K}{\hbar} \cos x\right) \exp\left(-i \frac{\hat{p}^2}{2\hbar}\right)$$

The wavefunction evolves from one kick to the next as

$$\psi(x, n+1) = \hat{U}\psi(x, n)$$

Angular Momentum Basis Representation

We can use Bessel functions to write the Floquet operator in angular momentum basis. The matrix elements will be:

$$U_{mn} = (-i)^{m-n} J_{m-n} \left(\frac{K}{\tilde{\hbar}} \right) \exp \left(-i \frac{\tilde{\hbar}}{2} n^2 \right)$$

This is a banded matrix as for $|m - n| \gg K/\tilde{\hbar}$, the Bessel function $J_{m-n} \rightarrow 0$.

Mean Energy of the System

In the classical regime, the mean square momentum $\langle p^2 \rangle$ grows linearly with time:

$$\langle p^2 \rangle \approx D(K)t$$

In the quantum regime, however, the energy growth saturates after a characteristic break time $t^* \sim K^2/\hbar^2$, leading to an exponential localisation in momentum space:

$$P(n) \sim \exp(-2|n|/L)$$

where L is the localisation length.

Classical vs Quantum Energy

We simulated the system normally and also with added noise to the kick strength to find out what would happen.

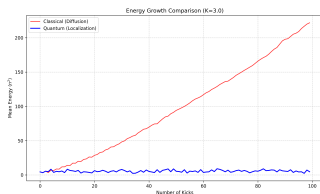


Figure: Classical vs Quantum Energy Growth

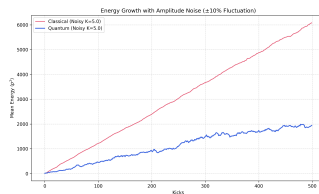


Figure: Classical vs Quantum Energy Growth with Noise in Kick Strength

As we can see noise disrupts the precise quantum interference required for localisation, causing the system to resume classical-like diffusive energy growth over long time scales.

Quantum Kicked Rotor on the Cylinder

The quantum kicked rotor is naturally defined on a cylindrical phase space,

$$\theta \in [0, 2\pi), \quad n \in \mathbb{Z}$$

with momentum eigenstates satisfying

$$\hat{p}|n\rangle = n\hbar_{\text{eff}}|n\rangle$$

Why Introduce the Torus?

The cylinder formulation has an infinite-dimensional Hilbert space:

$$n = -\infty, \dots, \infty$$

This is useful for studying dynamical localization, but it is not convenient for direct numerical studies of finite-dimensional Random Matrix Theory. We therefore impose periodic boundary conditions in both position and momentum.

Quantum Kicked Rotor on the Torus

The phase space becomes compact and the Hilbert space has a finite dimension N .

This gives a finite-dimensional Floquet operator, $\hat{U} \in U(N)$ which can be diagonalized numerically and compared directly with Circular Random Matrix Ensembles.

QKR on the Torus

For the torus formulation, the effective Planck constant is chosen as

$$\hbar_{\text{eff}} = \tau = \frac{4\pi}{N}$$

so that the quantum and classical phase spaces are compatible. The finite-dimensional Floquet operator is an $N \times N$ unitary matrix. Its eigenvalues therefore lie on the unit circle:

$$\hat{U}|\psi_\alpha\rangle = e^{-i\phi_\alpha}|\psi_\alpha\rangle$$

where ϕ_α are the eigenphases or quasienergies.

Floquet Spectrum

Unlike a time-independent Hamiltonian, the QKR is periodic in time.

Therefore, the relevant spectral quantities are the eigenphases

$$\phi_1, \phi_2, \dots, \phi_N$$

The eigenphases are defined modulo 2π because

$$e^{-i(\phi_\alpha + 2\pi)} = e^{-i\phi_\alpha}$$

Thus the spectrum naturally lives on a circle rather than on the real energy axis.

Finite-N Floquet Operator

The finite-dimensional Floquet operator introduced by Izrailev is employed.

In the momentum basis, the matrix elements are

$$U_{nm} = \frac{1}{N} e^{-i\tau n^2/4} \sum_{l=-N_1}^{N_1} \exp \left[-ik \cos \left(\frac{2\pi l}{N} + \theta_0 \right) \right] e^{-2\pi i l(n-m)/N} e^{-i\tau m^2/4}$$

where $N_1 = \frac{N-1}{2}$ $k = \frac{K}{\tau}$.

Symmetry and the Circular Ensembles

The parameter θ_0 controls the time-reversal symmetry of the Floquet operator.

Time-reversal symmetric

$$\theta_0 = 0$$

The statistics approach those of COE.

Time-reversal symmetry broken

$$\theta_0 \in \mathbb{R} \setminus \mathbb{Q}$$

The statistics approach those of CUE

Spacing Ratio Distribution

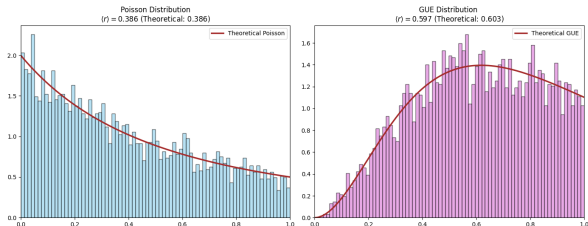


Figure: Spacing-ratio distribution for the quantum kicked rotor.

The numerical distribution can be compared with the analytical predictions of Random Matrix Theory.

Short Detour

What is the Spectral Form Factor?

The spectral form factor (SFF) measures correlations between eigenphases at different spectral scales.

For Floquet eigenphases

$$\{\phi_n\}_{n=1}^N, \quad U|\psi_n\rangle = e^{-i\phi_n}|\psi_n\rangle,$$

define the trace

$$Z(t) = \sum_{n=1}^N e^{-it\phi_n}$$

The spectral form factor is

$$K(t) = \langle |Z(t)|^2 \rangle = \left\langle \sum_{m,n} e^{-it(\phi_m - \phi_n)} \right\rangle$$

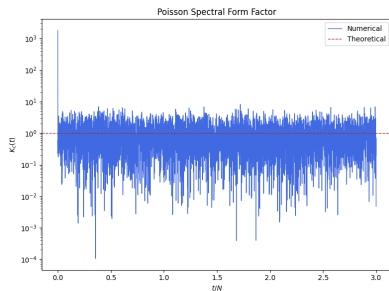
Dip–Ramp–Plateau

For chaotic quantum systems, the spectral form factor displays the following characteristic behaviour:

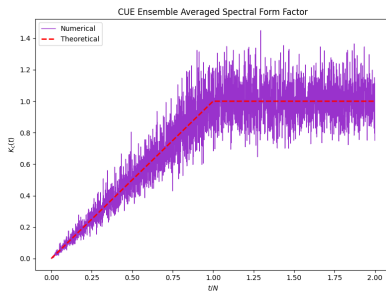
dip $\rightarrow$ ramp $\rightarrow$ plateau

- ▶ **Dip:** destructive interference between eigenphases.
- ▶ **Ramp:** long-range spectral correlations develop.
- ▶ **Plateau:** finite Hilbert-space dimension limits the correlations.

Spectral Form Factor of the QKR



(a) Cylinder



(b) Torus

Figure: Spectral form factor for the QKR in the two geometries.

Long-Range Statistics of the QKR on Cylinder

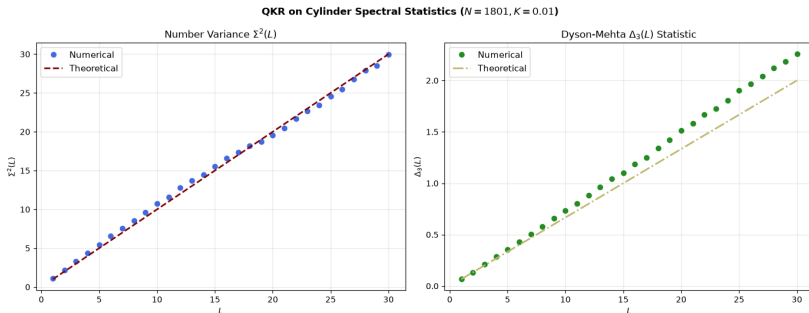


Figure: $\Sigma^2(L)$ and $\Delta_3(L)$ for Cylinder Geometry

Long-Range Statistics of the QKR on Torus

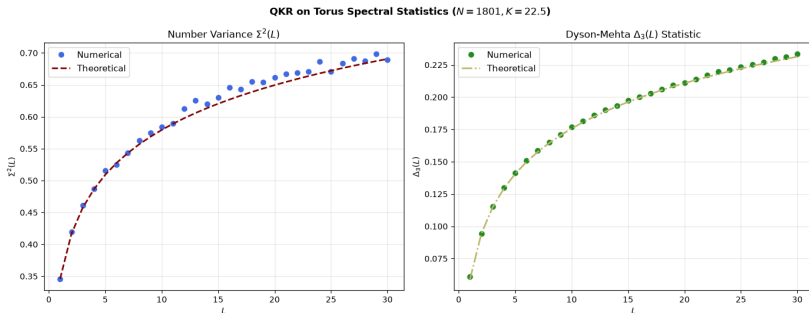


Figure: $\Sigma^2(L)$ and $\Delta_3(L)$ for Torus Geometry

Why Study Eigenvectors?

Eigenvalue statistics tell us about correlations between energy levels.

But they do not tell us how the corresponding eigenstates are distributed in Hilbert space.

For a Floquet eigenstate,

$$|\psi_\alpha\rangle = \sum_{n=1}^N c_n^{(\alpha)} |n\rangle$$

the coefficients $c_n^{(\alpha)}$ describe the distribution of the state in the momentum basis.

We can therefore ask:

How localized or ergodic are the Floquet eigenstates?

Porter–Thomas Distribution

For the CUE, the components of a Floquet eigenvector behave as complex Gaussian random variables.

Define the normalized intensity

$$x = N|c_n|^2$$

The universal CUE prediction is

$$P(x) = e^{-x}$$

which is the Porter–Thomas distribution for complex eigenvector components.

Porter–Thomas Distribution

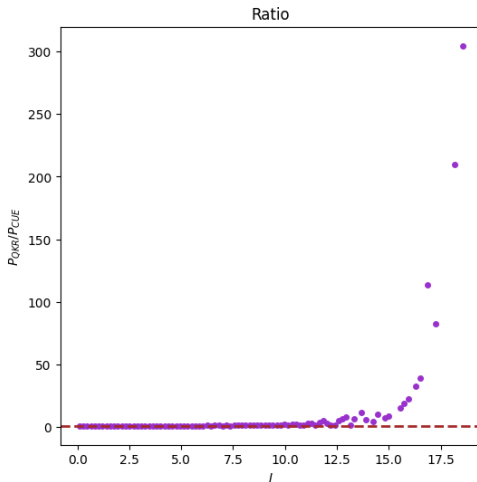


Figure: Normalized Floquet eigenvector intensities compared with the

Inverse Participation Ratio

For a normalized eigenstate, the inverse participation ratio is

$$\text{IPR} = \sum_{n=1}^N |c_n|^4$$

The IPR measures how concentrated the state is in the chosen basis.

- ▶ Completely localized state: $\text{IPR} = 1$
- ▶ Fully extended state: $\text{IPR} \sim \frac{1}{N}$

Therefore, smaller IPR corresponds to greater delocalization.

Participation Ratio

The participation ratio is the inverse of the effective number measured by the IPR:

$$\text{PR} = \frac{1}{N \text{ IPR}}$$

It is normalized between zero and one.

Thus, PR provides a convenient measure of how much of the available Hilbert space an eigenstate occupies.

Shannon Entropy

Another measure of eigenstate delocalization is the Shannon entropy,

$$S = - \sum_{n=1}^N |c_n|^2 \ln |c_n|^2$$

For a localized state, $S \approx 0$.

For an ergodic CUE eigenstate, $S \simeq \ln N - (1 - \gamma)$, where γ is the Euler–Mascheroni constant.

Generalized Participation Moments

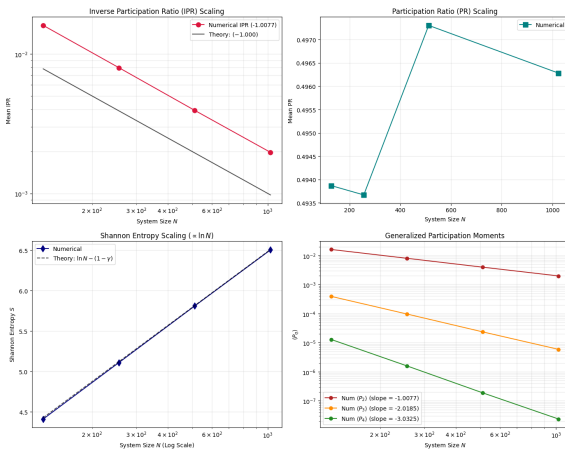
The localization properties can be generalized using

$$P_q = \sum_{n=1}^N |c_n|^{2q} \quad q > 1$$

For extended states, $P_q \propto N^{-(q-1)}$, whereas localized states remain approximately independent of N .

The scaling of P_q therefore provides a more detailed probe of ergodicity and possible multifractal behaviour.

Finite-size Scaling for QKR on Torus



Figure

Conclusion

So, what does chaos mean in quantum mechanics?

The classical picture of chaos does not carry over directly to quantum mechanics. Instead, chaos reveals itself through statistical and dynamical signatures.

- ▶ **Random Matrix Theory** provides universal predictions for spectral statistics of quantum chaotic systems.
- ▶ **Short-range statistics** such as level spacings and spacing ratios reveal level repulsion and distinguish Poisson from RMT-like spectra.
- ▶ **Long-range statistics** such as $\Sigma^2(L)$ and $\Delta_3(L)$ probe spectral rigidity and correlations.
- ▶ The **quantum kicked rotor** is one such system that connects classical chaos with quantum dynamics, showing dynamical localization despite classical momentum diffusion.

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